Violations of First Order Stochastic Dominance

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Abstract

I find necessary and sufficient conditions for first order stochastic dominance (FOSD) violations for choices from a budget line of Arrow securities. Applying this characterization to existing data, I compare FOSD violation rates across a broad set of risk preference elicitation tasks.[1]

Keywords: Stochastic Dominance, Risk Aversion, Experiment, Elicitation, Multiple Price List

JEL Classifications: C91, D81, D89

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1 Introduction

Through lottery decisions, economic agents can reveal their level of risk tolerance. Agents can, however, make decisions that are inconsistent with most classical decision theory, namely choices that are first-order stochastically dominated (FOSD). Such a choice is defined, roughly, as accepting a lesser prize or a lower probability of a higher prize.

Previous studies have investigated FOSD or inconsistent choice either as a necessity to explain subsets of their data (Holt and Laury (2002)) or to test the impacts of complexity in decisions under risk (Charness et al. (2007), Charness et al. (2018)). These studies span both the lab and the field (Jacobson and Petrie (2009), Galarza (2009)). Depending on the complexity of the lottery choice, task type and elicitation setting, FOSD violation rates (or inconsistent choices) have varied greatly across studies, ranging from under 10% to around 50%. The majority of these studies focus on a single decision or elicitation task type, often repeated with some slight variation in riskiness.

This paper contributes to these literatures by documenting the prevalence of stochastically dominated choices across several commonly used elicitation tasks in a single experiment. Theoretically, I provide the conditions for which a risky decision over Arrow securities along a budget line yields the possibility of a FOSD violation, while empirically, I check violation frequency in a set of important tasks against a pair of interesting benchmarks.

2 Data

The theoretical environment, experimental setting and data used in this report are from the recent risk elicitation paper Friedman et al. (2021) (henceforth referred to as VRE21). The experiment had 142 undergraduate students at UC Santa Cruz each engage with 56 risk elicitation trials using six different sorts of tasks. The design was entirely within-subject, with variation occurring in price and probability ordering, task block ordering, and within task block monotonicity/randomness. See VRE21 for a full characterization of the design.

In a given elicitation task, a subject chose a bundle (x, y) of Arrow securities; the bundle delivers x in state X (probability $\pi_X > 0$) and y in state Y (probability $\pi_Y = 1 - \pi_X > 0$). The x and y securities have prices of p_x and p_y, respectively. This means the agents solve the maximization problem

$$\max_{(x,y)} \pi_X u(x) + \pi_Y u(y) \quad st \;\; p_x x + p_y y = m$$

according to standard decision theory . The endowment m is set in each trial such that the corner bundle for the cheaper security holds 100 units of said security. Here $u(\cdot)$ is the agent's smooth, strictly increasing Bernoulli function, representing her preferences over the securities' payout.

After solving the first order conditions, VRE21 defined statistic L as the negative logarithm of the marginal rate of substitution[2]:

$$L \equiv ln\ \pi_X - ln\ \pi_Y - p_x + p_y$$

While L serves as the main regressor in VRE21's extraction of subjects' elicited risk aversion γ, this paper will use L for establishing a measure for violation severeness. Each trial seen by each subject can be associated with a single value of L.

Of the six sorts of tasks considered, five of them offer opportunities for FOSD violations: Holt-Laury, Budget Line, two variations of a new task named Budget Jars, and a spatial version of Holt-Laury named Budget Dots - Holt-Laury. The Holt-Laury (HL) task, originating from Holt and Laury (2002), is a text-based multiple price list which has 6 (traditionally 10) consecutive choices between two lotteries. The Budget Line (BL) task, per Choi et al. (2007), asks subjects to choose a bundle along a budget line (see Figure 1). Budget Jars, an elicitation task developed in VRE, has subjects begin with a "jar" of cash and use sliders to spend the cash on two Arrow securities, with (BJ) and without (BJn) cash retention allowed. The final task type, Budget Dots - Holt-Laury (BDHL), portrays each of the six lines of HL as a separate budget line, with the two feasible choices appearing as dots on the line.

3 FOSD Characterization

Suppose that $\pi_X = \pi_Y = 0.5$ and $p_x = 0.4$ while $p_y = 0.6$. No matter what her risk preferences, an agent facing these prices and probabilities should never choose a point on the budget line with $x < y$. For example, suppose she considered choosing $(x, y) = (7.5, 15)$, exhausting her budget $m = 12$. Since the states are equally likely, she'd be just as happy with $(15, 7.5)$, no matter what her Bernoulli function is. But the portfolio $(15, 7.5)$ costs only 10.5, so she could afford to spend 1.5 more on either Arrow security and be strictly better off than at $(x, y) = (7.5, 15)$.

The general result is expressed in terms of first order stochastic dominance (FOSD). Recall that lottery A (strictly) FOSDs lottery B iff $F_A(x) \leq F_B(x)$ for all x, with strict inequality for some x. The definition refers to the cumulative distribution function $F_Z(x)$, the probability that the realized payoff in lottery Z is no greater than x. Recall also (e.g., Mas-Colell, Whinston, and Green (1995), p. 195) that every expected utility maximizing agent prefers lottery A to B iff A FOSDs B.

Proposition 1 *A choice (x, y) on the budget line (1) is strictly first order stochastically dominated by another choice on the same budget line iff*

a. one Arrow state (e.g., X) is more likely and its security is less expensive (e.g., $\pi_X \geq \pi_Y$ and $p_x \leq p_y$), with at least one of these comparisons strict; and

b. the choice includes strictly less of the less-expensive-more-likely security (e.g., $x < y$).

See Appendix A for a proof, which can be generalized in a straightforward manner to cover Prospect Theory with symmetric probability weighting as well as Disappointment Aversion and some other generalizations of expected utility theory.

The Proposition tells us that every choice on the budget line can be rationalized by some Bernoulli function if the more likely state has a higher price, or if $L = 0$. But some choices will be dominated when prices are equal and probabilities differ, or the reverse, and when the more likely state has a lower price. In those cases, I can test for the rationality of subjects without committing to a functional form. For example, in Figure 1, the budget line crosses the diagonal at (400, 400)/9; any choice on the budget line with $x > 400/9$ is strictly dominated by an interval of choices with $x < 400/9$.

4 Empirical Results

Table 1 shows the overall frequency of dominated choices in our experiment. Multicrossings in 6-row HL or BDHL trials imply dominated choices (see Appendix C), and these appear in the Table's last three columns. The HL violation rate is 8.2%, which is slightly lower than those found in recent studies such as Charness et al. (2018), though the HL task in VRE21 yields fewer chances to multicross. BDHL follows relatively closely in both $p = 0.81$ trials (11.4% violation rate) and $p = 0.58$ trials (18.6%). The other columns report first order stochastic dominance violations in the remaining tasks, where Proposition 1 applies. A violation is deemed "major" if its log ratio lies outside the rectangular hyperbola $\ln(\frac{x}{y}) \cdot L = -1$. Table 1 shows a fair number of minor violations of FOSD, but rather few major violations. Table B.1 in Appendix B looks at tighter criteria for major violations, and confirms that a large majority of actual violations are small, due to clicking just a few dozen pixels away from an undominated choice in the BL task, or to purchasing just a little of an asset that is more expensive but not more likely in the BJ tasks. To summarize,

Result Dominated choices are uncommon in all tasks, and only about 1% of observations in relevant tasks are major violations of first order stochastic dominance (FOSD).

Additionally, I check these counts against two theoretical benchmarks. The first check makes use of a set of 1000 monte carlo trials simulated in the style of Apesteguia and Ballester (2018), run for and fully explained in VRE. I find that the human subjects violated more often than the simulated agents in the majority of investigated tasks (the opposite being true for the two Budget Jar tasks), yet committed major violations far less often than the sim agents. Uniform random choice serves as the other main benchmark. In all cases, the human subjects made violating choices far less often than agents choosing randomly.

5 Conclusion

I characterize FOSD violation in an important set of tasks. Using data from Friedman et al. (2021), I investigate FOSD violation rates across several elicitation methods. Violations are relatively uncommon, falling into the range generally seen in the literature, while major violations, are very rare across all task types studied. Human subjects make violations more often than Apesteguia and Ballester (2018) inspired simulated agents in most tasks, yet human-made violations are generally much less severe.

Notes

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[2]See Habib et al. (2017) for further discussion on L.

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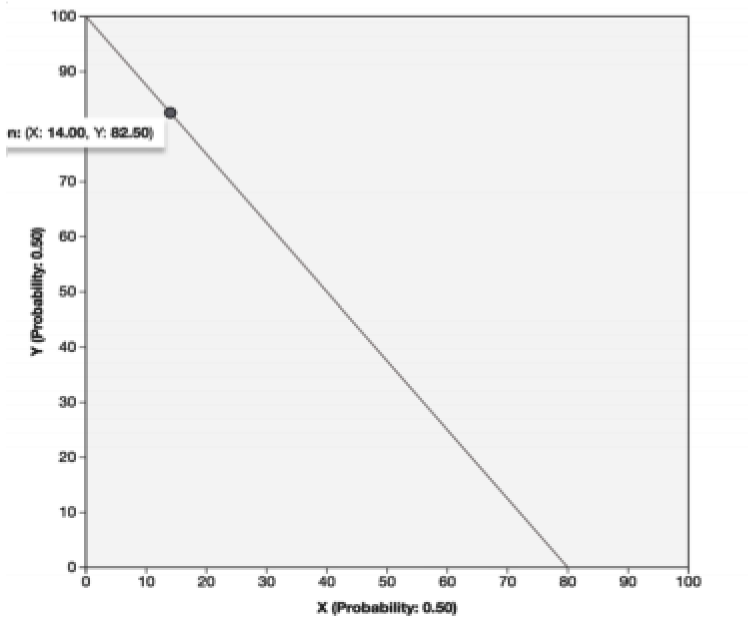

Figure 1: Budget Line task example (slope of 1.23, probability ratio of 1). Any choice of bundle with more of the expensive security (x here) is first-order stochastically dominated by any choice with the cheaper security holding the majority. Adapted from Friedman et al. (2021).

	BL	BJ	BJn	HL	BDHL (0.81)	BDHL (0.58)
Opportunities	1960	1278	1247	280	70	70
Violations	263	131	135	23	8	13
(Sim. Avg.)	141	197	146	13	6	6
(Sim. Perc.)	100	0	15	100	86	100
(Random)	761	497	484	253	63	63
Major Violations	17	6	16	-	-	-
(Sim. Avg.)	50	59	57	-	-	-
(Sim. Perc.)	0	0	0	-	-	-
(Random)	233	188	182	-	-	-

Table 1: Violations of FOSD. "Opportunities" is the number of trials for each task that allowed violations of FOSD. "Violations" is the number of such violations."(Sim. Avg.)" reports, to nearest integer, the average number of violations in each task across 1000 monte carlo simulations (see VRE21 for additional info). "(Sim. Perc.)" is the percentile the human data falls into within the 1000 trials. "(Random)" gives, to nearest integer, the expected number of violations given iid uniformly distributed random choices in each task. A violation (x, y) at L is deemed "major" if $L \cdot \ln(\frac{x}{y}) \leq -1$. Counts for 140 subjects used in VRE21 analysis (check VRE21 for subject drop explanation). Only half of the subject pool interacted with BDHL.

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Appendix A Proof of Proposition 1

A budget line is the set of lotteries $(x, y) \in \mathbb{R}^2$ satisfying $xp_x + yp_y = m$, where m is an (implicit or explicit) endowment of cash, and $p_x > 0$ and $p_y > 0$ are the prices of the two Arrow securities, with state probabilities $\pi_X, \pi_Y > 0$ and $\pi_X + \pi_Y = 1$.

Recall that a lottery L FOSD's another lottery M if their cumulative distribution functions (cdf's) satisfy $F_M(z) - F_L(z) \geq 0$ for all $z \in \mathbb{R}$, and that the lottery ordering is strict if the inequality is strict for some $z \in \mathbb{R}$.

Proposition 1 *A lottery $(x, y) \in \mathbb{R}^2$ is strictly first order stochastically dominated by another lottery on the same budget line iff*

a. one Arrow state is more likely and its security is less expensive (e.g., $\pi_X \geq \pi_Y$ and $p_x \leq p_y$), with at least one of these comparisons strict; and

b. the lottery includes strictly less of the less expensive security (e.g., $x < y$).

Proof. First consider the case $\pi_X \geq \pi_Y$ and $p_x < p_y$, and suppose that $x < y$. The cdf for lottery (x, y) is

$$\begin{aligned} F(z) &= 0 \quad \text{if} \quad z < x \\ &= \pi_X \quad \text{if} \quad x \leq z < y \\ &= 1 \quad \text{if} \quad z \geq y \end{aligned}$$

We will construct another lottery (a, b) on the same budget line as (x, y) in two steps, and show that it strictly FOSD's (x, y). First set $a = y$ and $b' = x$, and let G be its corresponding cdf. Then $F(z) - G(z) = 0$ for $z < x$ and $z > y$, but $F(z) - G(z) = \pi_X - \pi_Y \geq 0$ for $x \leq z < y$, so the lottery (a, b') weakly FOSD's (x, y). Now set $b = b' + c/p_y$, where

$c = (y-x)(p_y - p_x) > 0$ by hypothesis, and let H be the cdf for the lottery (a, b). Clearly $G(z) = H(z)$ except for $y < z \leq y + c/p_y$, where $G(z) - H(z) = 1 - \pi_X > 0$. Thus (a, b) strictly FOSD's (a, b') and thus, by transitivity, strictly FOSDs (x, y). To complete the proof for the present case we need only verify that the expenditure on (a, b) is the same as on (x, y):

$$ap_x + bp_y = yp_x + (x + c/p_y)p_y = yp_x + xp_y + c = yp_x + xp_y + (y-x)(p_y - p_x) = xp_x + yp_y = m.$$

The other cases have very similar proofs. For example, if $\pi_X > \pi_Y$ and $p_x \leq p_y$, then the conclusion follows from the fact that (a, b') strictly FOSD's (x, y). Of course, we can only guarantee weak FOSD of (x, y) with $y > x$ when both $\pi_X \geq \pi_Y$ and $p_x \leq p_y$. To show that (x, y) with $y < x$ is FOSD'd when $\pi_X \leq \pi_Y$ and $p_x \geq p_y$, we use precisely the same approach interchanging the roles of X and Y.

To complete the proof, we need only show that no lottery on the budget line is strictly FOSD'd when (i) $\pi_X > \pi_Y$ and $p_x > p_y$ or (ii) $\pi_X < \pi_Y$ and $p_x < p_y$, and to check subcases where the inequalities are weak. Of course, the arguments are the same for (ii) as for (i) due to the symmetric roles of X and Y, so it suffices to consider only case (i). For this case, let F, G be the cdfs for lotteries $(x, y) \neq (a, b)$ on the same budget line. Since the line is negatively sloped, one of the points, say (x, y), is northwest of the other, so $x < a$ and $b < y$. There are now three subcases.

1. Both points are above the diagonal $x' = y'$. Since $p_x > p_y$, we have $x < a < b < y$. It follows that $F(z) - G(z) = \pi_X > 0$ for $x \leq z < a$ but $F(z) - G(z) = \pi_X - 1 < 0$ for $b \leq z < y$. Hence neither point FOSD's the other.

2. Both points are below the diagonal $x' = y'$. Since $p_x > p_y$, we have $b < y < x < a$. It follows that $F(z) - G(z) = 0 - \pi_Y < 0$ for $b \leq z < y$ but $F(z) - G(z) = 1 - \pi_Y > 0$ for $x \leq z < a$; again, no FOSD ranking.

3. $x < y$ but $a > b$. We can not have $x < b < y < a$, as this would imply that the budget line has -slope $\frac{y-b}{a-x} < 1$ but the hypothesis $p_x > p_y$ implies -slope > 1. The other three orderings $b < x < a < y$, $b < x < y < a$ and $x < b < y < a$, are possible, but each

implies a change in the sign of $F(z) - G(z)$. For example, with $b < x < y < a$, we have $F(z) - G(z) = 0 - \pi_Y < 0$ for $b \leq z < x$ but $F(z) - G(z) = 1 - \pi_Y > 0$ for $y \leq z < a$.

The subcases where the inequalities are weak follow from taking limits as $\frac{p_x}{p_y} \to 1$ and $\frac{\pi_X}{\pi_Y} \to 1$.

□

Appendix B Additional Tables

B.1 Major Violation Cutoff Robustness

Table B.1 shows the progression of violations over a subset of cutoffs $c \in [-1, -0.05]$. As the criteria for major violations, $L \cdot \ln(\frac{x}{y}) \leq c$, weakens from -1 towards 0, the number of violations naturally increases. Even at $c = -0.05$, over half of the BL violations are still not considered major violations, indicating the majority of violations are from being only a handful of pixels away from what was likely intended to be a choice along $y = x$.

FOSD Major Violations Over Range of Cutoffs											
Major Cutoff c	-0.05	-0.1	-0.2	-0.3	-0.4	-0.5	-0.6	-0.7	-0.8	-0.9	-1
BL Major Violations	92	73	49	36	28	27	24	23	21	17	17
BL Major Random	715	646	5440	466	410	366	330	300	275	252	233
BJ Major Violations	57	38	21	17	15	12	8	7	7	6	6
BJ Major Random	477	445	392	350	317	288	263	241	222	204	188
BJn Major Violations	58	46	32	25	22	21	20	19	17	17	16
BJn Major Random	466	433	380	339	306	278	254	233	214	197	182

Table B.1: Major violation counts under a set of major cutoffs. Each column header is a different cutoff value c for the inequality $L \cdot \ln(\frac{x}{y}) \leq c$.

Appendix C HL FOSD Characterization

Take a trial of HL, where each row is a choice between two lotteries, A and B. Let lottery A be the safe lottery (closer to $y = x$) and B be the risky lottery (closer to a corner of the budget line) in each row. Each row of the trial can be characterized as follows: (x, y) @ (π_i^x, π_i^y) versus (x', y') @ (π_i^x, π_i^y), where π_i^j is the state probability for state j in row i. In the variants of HL used in this paper, the following hold: $x' > x$, $y' < y$, $x > y$, $\pi_i^x + \pi_i^y = 1$, $\pi_i^x < \pi_k^x$ for $i < k$, and $\pi_i^y > \pi_k^y$ for $i < k$.

Suppose a subject multicrosses, meaning B is chosen in some row m, while in some row $n > m$, A is chosen. **Note that each subject is assumed to have started with a choice of A. Even if in practice a subject selects B in row 1, he is assumed to have selected A in a preceding row had it been shown.** I conjecture that choosing A in row m and B in row n (call this choice AB) FOSDs choosing B in row m and A in row n (call this BA).

Assuming the set of row choices, not including rows m and n, in the two scenarios are the same, we can simplify the relevant payoffs for AB and BA such that row m and n will be chosen as the paying lottery with equal probability. Thus we can define the cumulative density functions for AB and BA, call them $F_{AB}(z)$ and $F_{BA}(z)$, as follows:

$$\begin{aligned} F_{AB}(z) &= 0 \quad \text{if} \quad z < y' \\ &= \frac{\pi_n^y}{2} \quad \text{if} \quad y' \leq z < y \\ &= \frac{\pi_n^y + \pi_m^y}{2} \quad \text{if} \quad y \leq z < x \\ &= \frac{\pi_n^y + \pi_m^y + \pi_m^x}{2} \quad \text{if} \quad x \leq z < x' \\ &= 1 \quad \text{if} \quad z \geq x' \end{aligned}$$

and

$$\begin{aligned} F_{BA}(z) &= 0 \quad \text{if} \quad z < y' \\ &= \frac{\pi_m^y}{2} \quad \text{if} \quad y' \leq z < y \\ &= \frac{\pi_m^y + \pi_n^y}{2} \quad \text{if} \quad y \leq z < x \\ &= \frac{\pi_m^y + \pi_n^y + \pi_n^x}{2} \quad \text{if} \quad x \leq z < x' \\ &= 1 \quad \text{if} \quad z \geq x' \end{aligned}$$

Thus we can see $F_{AB}(z) = F_{BA}(z)$ for all values of z except $z \in [y', y) \cup [x, x')$. Over this union, $F_{AB}(z) < F_{BA}(z)$ is clearly true, thus we have $F_{AB}(z) \leq F_{BA}$ $\forall$ $z \in \mathbb{R}$. By definition, AB FOSDs BA.

This sketch can be expanded to show more egregious multicrossings (more than two crosses) are also dominated by a reordering which forms a single crossing.