Linear pricing in double auction markets with non-convexities

Martin Bichler*, Vladimir Fux*, Jacob Goeree**
*Department of Informatics, TU München, Munich, Germany
**Department of Economics, University of Technology, Sydney, Australia

Electronic markets allow for efficient resource allocation, where this has not been possible only a few years ago. This paper is motivated by the design of a large-scale combinatorial exchange for fishery licenses in Australia. Similar to day-ahead energy markets, item-level and anonymous prices are very desirable in these markets. In general, a single linear price vector is only possible at the expense of social welfare. We introduce alternative ways to compute linear and anonymous prices, and analyze their impact on social welfare, the welfare distribution between buyers and sellers, and computational complexity. The payment rules include linear prices for the buyers only, for the sellers only, or linear price vectors for both parties. In numerical experiments based on field data, we show that the efficiency loss for some linear price vectors is negligible, but it is significant for a single linear price vector for both sides.

Key words: multi-object auctions, combinatorial exchange, payment rules

---

1. Introduction

Models of multi-object markets are a central topic in economics. For example, general equilibrium theory attempts to explain supply, demand, and prices in an economy with several interacting markets for objects. The celebrated Arrow–Debreu–McKenzie model suggests that under certain economic assumptions such as divisible objects, convex preferences, and demand independence there must be a set of item prices such that aggregate supplies will equal aggregate demands for every commodity in the economy (Arrow and Debreu 1954, McKenzie 1959). While some economists argue that these results are a good approximation at least for "large" markets where the non-convexities are "small" (Starr 1969), others contend that general equilibrium models have little connection to actual economies as their assumptions are almost never given in the field (Georgescu-Roegen 1979). In particular, goods are typically not divisible, and the resulting allocation problems are non-convex, which is often not considered in the general equilibrium literature (O'Neill et al. 2005).

In the recent years, the literature on market design emerged as an engineering discipline at the intersection of economics, computer science, operations research, and information systems aiming do design efficient market rules that satisfy real-world requirements.The design of allocation and payment rules for non-convex markets plays a central role in this literature.

This paper is motivated by the design of a real-world multi-object market with multiple buyers and sellers: a clearing house for fishery licenses (aka. shares) in Australia. The fishery license market consists of several regions, and in each region a number of different types of fishing licenses (aka. share classes) exists. Share classes describe the permission to catch a certain type and quantity of fish in a particular region, and each of the 100 share classes consists of a number of homogeneous shares describing units of effort, such as the number of hooks allowed for line fishing or the number of nets for net fishing, etc.

These shares are allocated inefficiently. There is a large percentage of around 1000 fishers who catch much less than what they are allowed to, while others would need additional shares. Bilateral bargaining among fishers turned out to be difficult due to the large number of fishers and share classes. As a consequence, the government initiated the design of a centralized exchange to allocate the shares efficiently. The design of such markets is challenging, because the allocation problem is a hard computational problem and because no payment rule can satisfy all desirable goals:

- The shares traded on the market are not divisible. Small fishers want to give up fishing and sell all their shares or nothing. Selling only a few shares would render their operation inefficient. Buyers are interested in buying certain quantities of shares from more than one class. Conceptually, this leads to a combinatorial exchange where sellers need the possibility to specify all-or-nothing sell-side package bids. While one-sided combinatorial auctions have been analyzed for several years, the literature on combinatorial exchanges is in its infancy. However, it is well known that the winner determination problem in a combinatorial exchange is NP-complete. Given the size of the market, it is unclear if the allocation problem can be solved to optimality.
- Mechanism design typically imposes incentive compatibility of a mechanism as a constraint. It is well-known that the Vickrey-Clarke-Groves (VCG) mechanism is the unique mechanism which exhibits dominant strategies (Green and Laffont 1977). Unfortunately, the VCG mechanism is not budget-balanced. Myerson and Satterthwaite (1983) showed that there exists no mechanism which is efficient, budget-balanced, individually rational, and Bayes-Nash incentive compatible simultaneously. Also, VCG payments are not in the core, i.e., there can be a group of losing bidders who, together with the auctioneer, can make themselves better off (Ausubel and Milgrom 2006). Finally, the VCG payments are not anonymous, i.e., bidders who buy (or sell) shares of a particular class can pay (receive) very different prices. This also holds for (Vickrey-nearest) core-prices as they are used for spectrum auctions nowadays (Bichler et al. 2013, Day and Cramton 2012, Day 2013).

1.1. Reasons for linear and anonymous prices

We will now discuss, why linear (i.e., item-level) and anonymous prices are desirable for an exchange, as compared to the non-linear and differential prices used in VCG or core-selecting auctions. The reasons address perceived fairness of the prices, arbitrage opportunities, and the lack of prices for packages nobody has bid on. The latter plays a role if the exchange prices should serve as a baseline for over-the-counter trades after the auction.

Let's first discuss *fairness* of prices. Recent experiences with core-selecting auctions for the sale of spectrum auctions suggest that such prices are considered unfair. For example, in Switzerland one of the bidders in a core-selecting combinatorial auction had to pay almost 482 million Swiss Francs and another bidder close to 360 million Francs although they won a similar set of licenses.[1] This can be explained by the fact that the marginal value of bidders to the social welfare is different, but it has become a substantial concern for bidders and auctioneers alike (Bichler et al. 2013, Janssen and Karamychev 2013, Levin and Skrzypacz 2014). Also in fishery license exchanges, it will be considered unfair that one seller gets a different price than another for the same shares, and it would lead to arbitrage opportunities as resale is allowed.

There are different notions of fairness. An auction allocation is called envy-free if every bidder likes his own bundle at least as well as that of anyone else in terms of payoff (Foley 1967). This requires to consider valuations and prices. However, the valuations of other market participants are unknown, while the prices on the market are public knowledge. We argue that weighted proportionality of prices according to the number of shares traded in each share class is a desirable fairness concept for our market, which is also accessible to an external observer. Anonymous and linear prices are proportional to the packages that bidders bought or sold. Hierarchical package bidding, an auction format used by the US Federal Communications Commission for the sale of spectrum licenses, used linear prices for this reason (Goeree and Holt 2010).

A disadvantage of VCG or non-linear and differential core prices are *arbitrage opportunities.* Let's assume resale is allowed after the auction as it is the case in our fishery market. Then one buyer with a low price could resell his share to another buyer, who got a high price and make an arbitrage gain. Similarly, a seller who sells his shares at a high price could use part of his revenue to buy the same shares again from buyers who paid a low price. As a consequence, the *law of one price* (LOP) is an economic concept which posits that an object must sell for the same price in all markets (Lamont and Thaler 2003). It is as important in our fishery markets as it is on financial markets with resale.

In addition, policy makers desire linear prices, because such prices can serve as a simple *baseline for the pricing* of various packages in over-the-counter trades of fishers after the combinatorial

[1] Results of the Swiss auction can be found at http://www.news.admin.ch/NSBSubscriber/message/attachments/26004.pdf.

exchange is closed. With 100 item prices it is simple to get an estimate for a package price for each of the 2^{165} possible packages. In a similar way, prices on day-ahead electricity markets serve as a baseline for long-term energy contracts.

1.2. On the impossibility of linear prices

In summary, there are a number of advantages of anonymous and linear prices for exchange markets such as our fishery auction: A linear *competitive equilibrium* price vector is such that all bidders maximize their payoff at the prices, and the auctioneer maximizes revenue. Per definition, such prices are *compatible with the allocation.* This means, losers know why they lost just by evaluating the prices, and winners know why they won. Anonymous and linear competitve equilibrium prices are also known as *Walrasian prices* reminiscent of the Walrasian auctioneer in economic theory. Unfortunately, it is not always possible to find Walrasian prices to support the efficient allocation, as the following example illustrates:

Example 1. Suppose there is a market with two identical items and two bidders. Bidder 1 has complementary valuations. He is only interested in the package of both items with a valuation of \$3, while bidder 2 wants either one or the other item with a value of \$2, but he is not willing to pay more for the package. Walrasian competitive equilibrium prices would need to be higher than \$2 for one item and less than \$3 for both items, which is impossible.

Gul and Stacchetti (1999) showed that if the bidders' utility functions satisfy the *gross substitutes condition* (Kelso and Crawford 1982), then a Walrasian (competitive) equilibrium exists. The condition says that if the price of item Y increases, then the demand for item X either remains constant or increases, but does not decrease. The condition does not allow for complements. Milgrom (2000) shows that an economy may not have a Walrasian equilibrium if only one agent has preferences that are complementary and all the others have preferences that satisfy the gross substitutes condition. Unfortunately, the sell-side package bids in the fishery license exchange are complements, because a fisher would only sell one share if he can also sell the others in the package.

In assignment markets, where each bidder wins at most one item, the solution to the winner determination problem, the assignment problem, is always integral, and the dual variables for the market clearing constraints can be interpreted as market prices (Shapley and Shubik 1971). More generally, Scarf (1990) pointed out that in markets with convexity assumptions, the simplex method is an effective device to derive equilibrium prices from the underlying linear program. Since the winner determination problem in combinatorial exchanges which allow for package bids is a (non-convex) integer program, we know that in general such linear prices are not possible (Gomory and Baumol 1960), such that Scarf (1994) suggested approximations of the linear prices.

The problem of finding market clearing prices in markets with non-convexities has attracted renewed interest due to block orders in energy markets. Electricity production exhibits substantial non-convexities due to start-up costs and minimum power output of power plants. Non-convexities also exist on the buy-side, because bidders are interested to win a certain quantity of energy several hours in a row. This is why day-ahead energy markets allow for package bids of multiple hours.

Due to the impossibility of linear prices in the optimal solution, there are two approaches. In the USA, markets implement the welfare-maximizing solution but deviate from linear prices by using side-payments, so called uplifts. Due to the non-convexities, there will be some accepted bids, which should not be accepted subject to the prices. Such problematic orders are financially compensated via so called uplifts. Unfortunately, such uplifts might exceed the welfare generated in the market (Van Vyve et al. 2011).

In Europe, exchanges implement linear prices and accept inefficient allocations in terms of welfare (Van Vyve et al. 2011): all electricity traded at the same time and location is traded at the same price. Such market prices are not a Walrasian equilibrium, because there are bids that are rejected although they could increase welfare (Meeus et al. 2009). There were a number of proposals how to compute or approximate linear prices on energy markets (Hogan and Ring 2003, O'Neill et al. 2005, Van Vyve et al. 2011, Martin et al. 2014). Some of them are based on the duals of the linear programming relaxation, others define a separate problem to minimize the uplift.

While day-ahead energy markets also use linear and anonymous prices, the requirements for a fishery license exchange are quite different leading to another winner determination problem, but also to different ways how linear prices can be computed.

1.3. Contributions

In this paper we propose alternative ways to compute linear and anonymous prices in a combinatorial exchange for fishery licenses and analyze their impact on social welfare.

- First, we suggest an allocation rule that computes a linear and anonymous price vectors for buyers, for sellers, and for both parties. The impact of a single linear price vector which supports the allocation for all participants on social welfare is substantial, which is why we also analyze rules where one side of the market has a pay-as-bid rule. We introduce mathematical programs to determine allocations and prices.

- Second, in a numerical simulation based on field data from an Australian fishery market we compare the different payment rules with respect to their impact on overall welfare, welfare distribution, and the empirical hardness of the resulting optimization problems.

In the next section, we introduce the bidding language and the winner determination problem. In section 3, we introduce different linear and anonymous payment rules. Section 4 provides the results of numerical experiments, before we conclude in section 5.

(a) Sell-side bid (b) Buy-side bid

Figure 1 Examples for buy-side and sell-side bids

2. The winner determination problem

Due to the large number of fishers participating the exchange is organized as a sealed-bid auction, a clearinghouse where allocation and prices are computed in one shot. We will first discuss the bid language, before we introduce the basic winner determination problem.

2.1. Bid language

It is important to provide participants with a bid language that lets them express their preferences adequately with a low number of parameters. Also, the bid language should be such that it permits solving the allocation and pricing problem to optimality. Buyers and sellers have different requirements for the bid language. Sellers are largely small fishers who intend to retire. They need to be able to specify all-or-nothing bids. Such a fisher typically has only one bid to submit, which includes all his holdings in various share classes. The bid has a single ask price for the whole set of shares, and represents how much at least the seller wants to get for his holdings. Selling part of his holdings would not be an option as it might render his operations unprofitable. Figure 1a illustrates an all-or-nothing sell-side bid.

Buyers want to win shares in one or more share classes. Buyers can submit several bids on multiple shares in a share class. We do not allow buyers to submit package bids across multiple share classes, because complementarities across classes are less of a concern. The absence of package bids on the buy side makes it easier for the auctioneer to match supply and demand, and the simple OR-language only requires buyers to list share classes of interest without having to submit bids on many possible packages of interest. The missing bids problem is a well-known issue in combinatorial auctions (Bichler et al. 2014).

Buyers can bid not for a specific number of units in each endorsements, but bid a linear price for a quantity interval. For example, a fisher may want to buy shares from share class A. He needs at least 125 units and at most 250 units, and is willing to pay up to 3$ for each unit. Figure 1b shows an example of a buyer bid.

2.2. The winner determination problem

Let us now introduce some necessary notation to formulate the winner determination problem. We have a set of share classes $\mathcal{L}$ (aka. endorsements), indexed by l. We consider a set of auction participants $\mathcal{I}$, which consist of $\mathcal{I}_S$ of sellers and $\mathcal{I}_B$ of buyers with $\mathcal{I}_B \cap \mathcal{I}_S = \emptyset$. We will relax this condition later and also allow bidders to either sell or buy shares. Each seller wants to sell a certain set of units in a share class and submits a single bid $s \in \mathcal{S}$ with $\mathcal{S}$ being the set of all seller bids. For now, we assume each seller $s \in S$ submits only one sell-side bid. The bid corresponds to the bidding language described in the previous section; a seller bid is a tuple $s = < Q_s^1, \dots, Q_s^{|\mathcal{L}|}, A_s >$, in which he specifies the number of units Q_s^l he wants to sell in share class $l \in \mathcal{L}$ together with an ask price A_s.

Each buyer can submit multiple bids for different share classes and every combination of these bids can become winning. Each bid $b \in \mathcal{B}$ is a tuple $b = < l_b, \overline{Y}_b, \underline{Y}_b, D_b >$, where $l_b \in \mathcal{L}$ is an endorsement for which the bid applies, $\overline{Y}_b, \underline{Y}_b$ are lower and upper bounds on the number of desired units in the corresponding endorsement l_b and D_b is bid price per unit in this share class within the bounds specified. Sometimes, we will write $b_i \in \mathcal{B}$ to denote one of the bids of a buyer $i \in B$.

An allocation is a vector $a = (x, y) \in R^{|\mathcal{I}|}$, where $x = \{x_s\}_{s \in \mathcal{S}}$ is the vector of seller bids when bid s is accepted such that $x_s = 1$ and $x_s = 0$ otherwise, and $y = \{y_b\}_{b \in \mathcal{S}}$ is the vector buyers' bids in the allocation with $y_b \in Z^+$ being the number of units allocated to a buy-side bid b. The allocation is feasible if for every winning buy-side bid $b \in \mathcal{W}_\mathcal{B}$, the number of units allocated lies in the interval $[\underline{Y}_b, \overline{Y}_b]$, and in each endorsement the number of units sold is greater or equal than number of units bought (supply-demand constraint).

Definition 1. An efficient allocation is a feasible allocation, which maximizes the sum of agent valuations, i.e., it solves the following problem:

$$
\begin{aligned}
\max \sum_{s \in \mathcal{S}} -A_s x_s + \sum_{b \in \mathcal{B}} y_b D_b & \qquad \text{(Efficient allocation)} \\
\text{s.t.} \sum_{s \in \mathcal{S}} -Q_s^l x_s + \sum_{b \in \mathcal{B}: l_b = l} y_b \leq 0, & \quad \forall l \in \mathcal{L} \\
y_b \in [\underline{Y}_b, \overline{Y}_b] \cup \{0\}, x_s \in \{0,1\}, \ y_b \in Z^+ &
\end{aligned}
$$

We can now formulate the efficient allocation problem as an integer program:

$$
\begin{aligned}
\max \sum_{s \in \mathcal{S}} -A_s x_s + \sum_{b \in \mathcal{B}} y_b D_b & \qquad \text{(WDP)} \\
\text{s.t.} \sum_{s \in \mathcal{S}} -Q_s^l x_s + \sum_{b \in \mathcal{B}: l_b = l} y_b \leq 0, \quad \forall l \in \mathcal{L} & \qquad (1) \\
y_b \leq \overline{Y}_b, \quad \forall b \in \mathcal{B} & \qquad (2) \\
y_b \leq M z_b, \quad \forall b \in \mathcal{B} & \qquad (3)
\end{aligned}
$$

$$\underline{Y}_b - y_b \leq M(1 - z_b), \quad \forall b \in \mathcal{B} \tag{4}$$
$$x_s \in \{0,1\},\ y_b \in Z^+, z_b \in \{0,1\}$$

Here constraint (1) ensures that number of licenses bought in each share class is less or equal than number of licenses sold (supply-demand constraint). Note that this equilibrium constraint does not require strict equality. In other words, some shares might be sold in a package, although only a subset of the shares in the package are assigned to buyers. This is desirable for the government who wants to buy out sellers, but different to day-ahead energy markets where an hour of energy must not be wasted. Constraints (2)-(4) states that number of licenses allocated to each buy-bid $b \in \mathcal{B}$ is either 0 or belongs to range $[\underline{Y}_b, \overline{Y}_b]$. This program describes the basic winner determination problem which we will refer to as WDP. It is straightforward to extend the bid language with features such as exit-or-stay bids: A fisher in a new set $\mathcal{I}_E$ wants to either buy new shares or sell all his shares as a package.

$$\underline{Y}^l_j x_i \leq y^l_{j:i \in \mathcal{I}_E} \leq \bar{Y}^l_j x_i \; \forall l \in \mathcal{L}, i \in \mathcal{I}_E \tag{5}$$

For the remainder, we will ignore exit-or-stay bids and focus on the WDP introduced above.

3. Payment rules

There are a number of possible payment rules. We will first start with non-linear and personalized VCG payments and use these as a baseline in our numerical simulations to compare against payments based on linear price vectors. Then we will discuss core-selecting payment rules, before we analyze different types of anonymous and linear prices.

3.1. VCG payments

We assume that all agents have quasi-linear utility, which means that $u_s(a, \pi_s) = \pi_s - A_s$ for sellers and $u_i(a, \pi_i) = \sum_{b_i : i \in \mathcal{I}_B} D_{b_i} y_{b_i} - \pi_i$ for buyers, where $a = (x, y)$ is the allocation and π_s and π_i is the total payment for a seller s and buyer i, resp. The Vickrey-Clarke-Groves (VCG) mechanism is the only efficient mechanism which is dominant strategy incentive compatible with such quasi-linear utilities.

Let us describe $w(\mathcal{B} \cup \mathcal{S})$ as the social welfare (i.e., the objective function value of the WDP) with the entire set of bids $\mathcal{B} \cup \mathcal{S}$, and $w(\mathcal{B} \cup \mathcal{S}/\{s\})$ the social welfare when seller s is excluded. Then, the VCG payments can be defined in the following way, where $\mathcal{W}_S$ is the set of winning sellers:

$$p_s^{\text{VCG}} = A_s + (w(\mathcal{B} \cup \mathcal{S}) - w(\mathcal{B} \cup \mathcal{S}/\{s\})),\ s \in \mathcal{W}_\mathcal{S}$$

VCG payments of buyers are computed analogously by removing all bids of a winning buyer to compute his marginal contribution to the social welfare. Apart from the VCG payments, we will also compute core-selecting payments with and without a budget balance constraint. We omit a detailed description of these payment rules due to space constraints and refer the interested reader to Day (2013).

3.2. Anonymous and linear prices

We have discussed in the introduction that linear and anonymous prices are not always possible with general valuations unless we sacrifice efficiency. If we compute a single linear price vector, which is compatible with the allocation, then there are no possibilities for arbitrage. However, also the impact on efficiency is substantial. Therefore, we also analyze relaxed forms of linear prices such as linear prices for buyers only or linear prices for sellers only. This allows us to evaluate the trade-offs between stronger forms of linear prices and social welfare loss.

3.2.1. One linear price vector for sellers only (seller-linear, SL) We will start with the simplest form of linear prices, a price vector for the sellers only, where the buyers submit pay-as-bid prices. Such prices can be computed by extending the WDP via additional constraints.

$$\text{extend WDP} \tag{SL WDP}$$

$$\sum_{s\in\mathcal{S}}\sum_{l\in\mathcal{L}} -Q_s^l\lambda_s^l + \sum_{b\in\mathcal{B}} y_b D_b \geq 0 \tag{BB}$$

$$\sum_{l\in\mathcal{L}} \lambda_s^l Q_s^l \geq A_s x_s \; \forall s\in\mathcal{S} \tag{6}$$

$$\lambda_s^l \leq M x_s, \;\; \forall s\in\mathcal{S}, l\in\mathcal{L} \tag{7}$$

$$\lambda_s^l \leq p^l, \;\; \forall s\in\mathcal{S}, l\in\mathcal{L} \tag{8}$$

$$\lambda_s^l \geq p^l - (1-x_s)\overline{p}^l, \;\; \forall s\in\mathcal{S}, l\in\mathcal{L} \tag{9}$$

$$\lambda_s^l \geq 0, p^l \geq 0$$

First, have new budget-balance constraint (BB), which makes sure that the auctioneer does not make a loss (6). Then, constraints (7)-(9) linearizes the variable multiplication $x_s p^l$. $\overline{p}^l$ describes an upper bound for the linear price per license l, while M is a big number.

3.2.2. One linear price vector for buyers only (buyer-linear, BL) An alternative is to compute linear prices for buyers only and have sellers pay what they bid.

$$\text{extend WDP} \tag{BL WDP}$$

$$\sum_{s\in\mathcal{S}} A_s x_s + \sum_{b\in\mathcal{B}} \sum_{h\in\mathcal{B}:l_h=l_b} D_h \sigma_{j,h} \geq 0 \tag{BB}$$

$$\sum_{b\in\mathcal{B}:l_b=l_h} D_b u_b \leq r_h D_h + M(1-r_h), \quad \forall l\in\mathcal{L}, \forall h\in\mathcal{B}: l_h = l \tag{10}$$

$$y_b \leq \bar{Y}_b r_b, \quad \forall b\in\mathcal{B} \tag{11}$$

$$r_b \leq y_b, \quad \forall b\in\mathcal{B} \tag{12}$$

$$u_b \leq r_b \tag{13}$$

$$\sum_{b\in\mathcal{B}:l_b=l} u_b \leq 1, \quad \forall l\in\mathcal{L} \tag{14}$$

$$\sigma_{b,h} \leq y_b, \quad \forall b,h\in\mathcal{B}: l_h = l_b \tag{15}$$

$$\sigma_{b,h} \leq \bar{Y}_b u_h, \quad \forall b,h\in\mathcal{B}: l_h = l_b \tag{16}$$

$$\sigma_{b,h} \geq y_b - (1-u_h)\bar{Y}_b, \quad \forall b,h\in\mathcal{B}: l_h = l_b \tag{17}$$

$$\sigma_{b,h} \in Z^+; u_b, r_b \in \{0,1\}$$

The constraints (15)-(17) linearize the variable product $y_b u_h$ to $\sigma_{b,h}$. The payment for each bid is the sum of payments which one should pay if another participant's price was chosen as the linear price, and at most one $D_h\sigma_{b,h}$ product is positive for each $b\in\mathcal{B}$. The variable $u_b = 1$ if the corresponding bid b is the one with the lowest bid price D_b, among the accepted bids with $y_b > 0$. Constraint (10) and constraint (14) set variable $u_b^l = 1$ for one of the bids. Variable $r_b = 1$ if and only if $y_b > 0$, which is determined in constraints (11)-(12).

In all other linear pricing rules, we can enforce strict budget balance. In BL WDP the constraint (BB) cannot be modified to achieve strict budget balance without a risk to cause infeasibility. This is because no set of ask prices might exactly match particular set of bid prices in the winning allocation. A simple alternative is to compute the allocation with BL WDP, fix the allocation $a^* = (x^*, y^*)$ and then recompute the prices by replacing $\sum_{b\in\mathcal{B}}\sum_{h\in\mathcal{B}:l_h=l_b} D_h\sigma_{j,h}$ with $\sum_{b\in\mathcal{B}:l\in\mathcal{L}} p_l y_b^*$, where p_l is the price a buyer has to pay. Constraints (11)-(17) can now be removed, but one needs to restrict prices to be lower or equal to the winning bids: $p_l y_b^* \leq D_b y_b^*$.

3.2.3. Two linear price vectors for buyers and sellers (buyer-seller-linear, BSL) Another extension is to define two different linear price vectors, one for the sell and one for the buy side by adding the following constraint:

$$\text{extend the BL \& SL WDPs} \tag{BSL WDP}$$

$$\sum_{s \in \mathcal{S}} \sum_{l \in \mathcal{L}} -Q_s^l \lambda_s^l + \sum_{b \in \mathcal{B}} \sum_{h \in \mathcal{B}: l_h = l_b} D_h \sigma_{b,h} \geq 0 \tag{BB}$$

The budget-balance constraint (BB) now takes into account linear prices for the buy-side and the sell-side.

3.2.4. A single linear price vector (single-linear, 1L) With two more constraints, we can enable a single linear price vector for both, the buyers and the sellers.

$$\text{extend BSL WDP} \tag{1L WDP}$$

$$p^l = \sum_{b \in \mathcal{B}: l_b = l} D_b u_b^l, \forall l \in \mathcal{L} \tag{singlePrice}$$

$$\text{s.t.} \sum_{s \in \mathcal{S}} -Q_s^l x_s + \sum_{b \in \mathcal{B}: l_b = l} y_b = 0, \quad \forall l \in \mathcal{L} \tag{18}$$

With a single vector of linear prices, we need supply to equal demand for the exchange to be budget balanced. This is why constraint 18 describes a strict budget-balance.

3.3. Allocation compatible single linear prices (C1L)

Even with a single linear price vector for buyers and sellers in the previous subsection it can happen that some bidders should be winning according to these prices, but they are not. In other words, the prices might not be compatible with the allocation for losing bidders. If the auctioneer wants to avoid such situations, he can include further constraints:

$$\text{extend BSL WDP} \tag{C1L WDP}$$

$$\sum_{b \in \mathcal{B}: l_b = l_h} D_b u_b^l \geq (1 - r_h) D_h - M r_h - M(1 - \sum_{b \in \mathcal{B}: l_b = l_h} u_b^l), \quad \forall l \in \mathcal{L}, \forall h \in \mathcal{B} \tag{19}$$

$$\sum_{l \in \mathcal{L}} \lambda_s^l Q_s^l \leq A_s(1 - x_s) + M x_s, \quad \forall s \in \mathcal{S} \tag{20}$$

This is the strongest form of linear prices. In contrast to energy markets, we have lower and upper bounds on the quantity of the buy-side bids, and we only have package bids on the sell-side. As a consequence the C1L WDP is often infeasible, because there is no set of prices such that all losing sell-side package bids are higher and all losing buy-side bids are lower than these prices. In our experimental evaluation, we do not consider C1L for this reason.

4. Experimental evaluation

In this section we will briefly describe experiments, which were designed to analyze the impact of different linear pricing rules on efficiency and on the distribution of buyer and seller surplus.

4.1. Bid generation

We generate bids for the simulation based on field data containing current possessions together with profit of slightly more than 1000 fishers, who are supposed to participate in the exchange. This data provides valuable information about the profitability of fishers businesses and allows us to derive various scenarios with fishers willing to quit business and sell their current shares as a package.

Our bid generation algorithm consist of two parts: the *package generator* decides which of the fishers participates as buyer, as sellers, or who does not participate at all. The *valuation generator* estimates the value of a certain package (or individual share class) for a fisher. The package generator randomly chooses fishers to participate in the auction via a participation probability ρ. It first computes the efficiency of a fisher as a ratio between the median income of all fishers and the income of a fisher from all his shares: the smaller the ratio, the higher the probability of a fisher to submit a sell-side bid. Based on a threshold for efficiency we decide on a fisher to become a buyer.

For each sell-side bid we check if the current fisher has profit from his business. If not, then we consider this fisher to be willing to quit the market and construct a package containing all shares he currently possesses. If fisher has positive revenue, he still may want to sell unused licenses. Hence, we include all licenses with low profit in the package.

Each buyer can submit multi-unit bids for different share classes. Here we consider two scenarios: 1) the fisher has highly profitable licenses and wants to acquire additional units in this share class, and 2) the fisher wants to purchase fishing rights in an adjacent region. For the first scenario we retrieve all share classes with revenue per catch being higher than the mean revenue for all share classes, and then construct a bid with a random number of units below the current number of units the fisher has in the share class. For the second scenario, we analyze all profitable share classes of a fisher has and select adjacent high-revenue share classes he may be interested in.

The valuations generator assumes that each share class has some common value (which depends on how profitable this class is in average among all fishers), and then each fisher can over- or under-estimate this common value. For this purpose we use two normal distributions for sellers and buyers, which are parametrized by a mean value μ (which is based on the common value) and standard deviation σ. We will further denote the relative difference between buyers and sellers valuation means by a spread Δ_μ. This simple setting allows us to model various valuation scenarios. For example, we can simulate markets where buy and sell-side valuations differ (which implies lower surplus) or where they are alike. In our experiments, we assume bidders submit their valuations truthfully to allow for a comparison of different payment rules.

4.2. Empirical hardness of the winner determination problem

The winner determination problem in our combinatorial exchanges is an NP-complete problem. It is therefore not obvious that relevant problem sizes can be solved to optimality. We first evaluated the time to solve the winner determination problem for the efficient allocation for different payment rules. Due to our bid language, even large scenarios can be solved in less than a minute to optimality (in average) on a laptop with Intel Core i7-4712HQ processor and 16.0 GB of RAM. For all numerical experiments we used a branch-and-cut implementation of the Gurobi mixed-integer programming solver version 6.0. Figure 2 shows the average results of 30 simulations for each problem size, which is determined by the participation probability ρ. The higher this probability, the more of the 1000 fishers are participating.

Figure 2 illustrates that we can solve realistic instances (with participation probability $\rho = 0.7$ we have nearly 900 bids). We also may notice strong impact of buyer-linear prices constraints on computation time: buyer-linear (BL) and buyer-seller linear (2L) required more time to solve than single linear and seller-linear prices. Note, that even if our formulation of single prices (1L) extends BL prices, we implemented it in a more efficient way, which explains lower runtime.

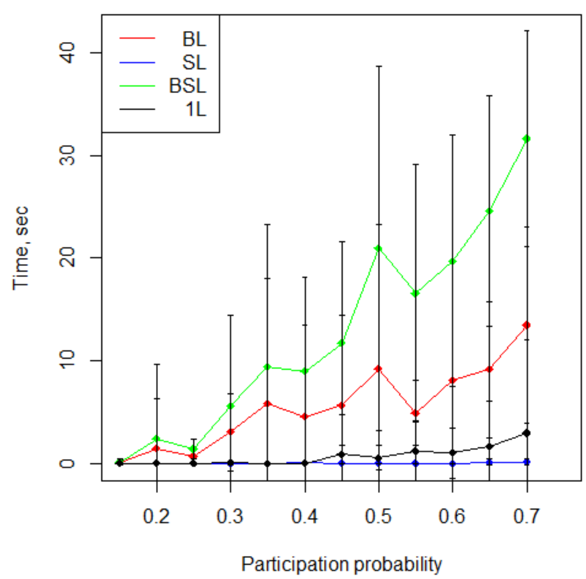

Figure 2 **Average computation time for different payment rules,** $\Delta_\mu = 0.4$**,** $\sigma = 0.2$

4.3. Welfare losses

Next, we analyze welfare losses due to linear prices. Figure 3 describes the efficiency losses, as percentage of the social welfare in efficient allocation, when we again vary participation probability.

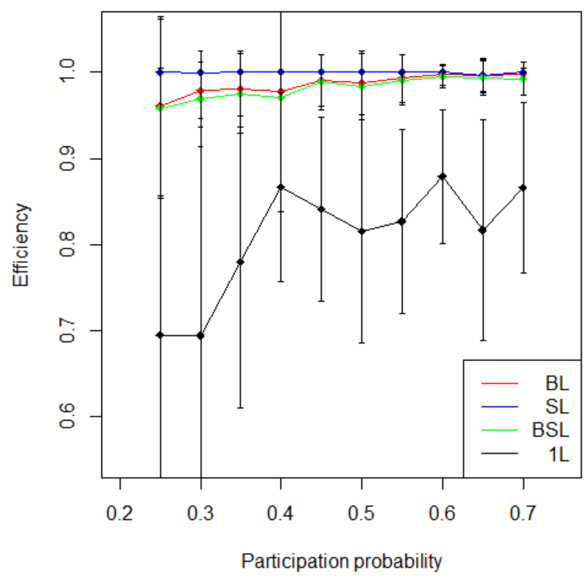

Figure 3 **Efficiency loss,** $\Delta_\mu = 0.4$**,** $\sigma = 0.2$

The curves presents the average results of 30 experiments on randomly generated bid instances for each value of participation probability ρ. Interestingly, single linear prices (1L) lead to significant welfare losses, while the two buyer-seller linear prices lead to very little efficiency losses. We conjecture, that the discounts bidders receive in these payment rules would already set incentives for truthful bidding. With pay-as-bid rules, we conjecture that bidders would shade their bids by some profit margin, even if they know little about their competitors.

4.4. Social welfare distribution

Finally, we compare how different payment rules distribute the social welfare among the participants. Note, that for all linear pricing rules we impose a strict budget balance constraint such that the auctioneer gets zero revenue. We have run the simulations with various Δ_μ values from a set $\{0.3, 0.5, 0.7, 0.9\}$ and computed the average payoffs for sellers and buyers. The VCG mechanism leads to a substantial budget deficit for the auctioneer, as Figure 4 illustrates. Note that neither BSL nor SL exhibit an efficiency loss, and the social welfare equals the one of the VCG mechanism reduced by the budget deficit. In other words, the auctioneer provides additional money, which is why the welfare of buyers and sellers increases.

The single linear price vector (1L) incurs a welfare loss. However, for both, 1L and BSL the welfare is distributed between buyers and sellers. This is different to seller-linear prices (SL), where the welfare goes to the sellers only. Consider that we do not take into account bid shading of buyers, which can be expected with a pay-as-bid rule.

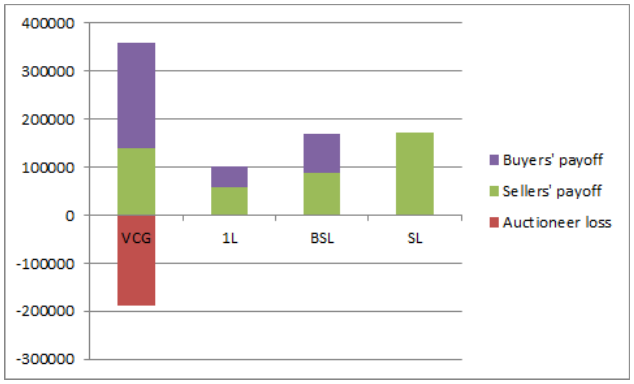

Figure 4 **Payoff distribution,** $\rho = 0.3$, $\sigma = 0.3$

5. Conclusions

Table 1 describes basic trade-offs between different linear anonymous and non-linear differential payment rules. The outcome of the VCG and Core-selecting auctions are fully efficient with truthful bidding, however, they are not budget balanced. If an auctioneer is willing to give up budget balance, also the outcome of BSL, SL, and BL is fully efficient. Therefore, the auctioneer needs to make a decision between budget balance and full efficiency in all payment rules.

Table 1 **Overview of payment rules for combinatorial exchanges**

	VCG	Core	C1L	1L	BSL	SL	BL
Efficiency (w. truthfulness)	✓	✓	✗	✗	✗	✗	✗
Budget balance	✗	✗	✓	✓	✓	✓	✓
Strategy-proofness	✓	✗	✗	✗	✗	✗	✗
No arbitrage opportunity	✗	✗	✓	❍	✗	✗	✗
Proportional fairness	✗	✗	✓	✓	✓	❍	❍

Neither core-selecting auctions nor linear price auctions are strategy-proof. However, the possibilities for profitable manipulation are much reduced in a fishery license market where there are large numbers of participants and items and there is little or no common prior information about the valuations of competitors.

Also if we assume that bidders submit their values truthfully, there is a trade-off between efficiency and arbitrage opportunities in different linear payment rules. A compatible linear price vector (C1L) is arbitrage-free, but one could argue that possibilities for arbitrage are also much reduced in 1L and BSL. Linear prices for both sides can be considered fair. Proportional fairness, might be challenged in SL and BL, but in particular in VCG and core-selecting payment rules where participants with the same allocation can receive different payments. Overall, there is a trade-off between the five different design desiderata and it is important for a regulator to make an informed decision.

References

Adomavicius, G., S. Curley, A. Gupta, P. Sanyal. 2013. Impact of information feedback in continuous combinatorial auctions: An experimental study of economic performance. *MIS Quarterly* **37** 55–76.

Arrow, Kenneth J, Gerard Debreu. 1954. Existence of an equilibrium for a competitive economy. *Econometrica: Journal of the Econometric Society* 265–290.

Ausubel, L., P. Milgrom. 2006. The lovely but lonely vickrey auction. P. Cramton, Y. Shoham, R. Steinberg, eds., *Combinatorial Auctions*, chap. 1. MIT Press, Cambridge, MA, 17–40.

Bichler, M., A. Gupta, W. Ketter. 2010. Designing smart markets. *Information Systems Research* **21**(4) 688–699.

Bichler, Martin, Jacob Goeree, Stefan Mayer, Pasha Shabalin. 2014. Spectrum auction design: Simple auctions for complex sales. *Telecommunications Policy* **38**(7) 613–622.

Bichler, Martin, Pasha Shabalin, Jürgen Wolf. 2013. Do core-selecting combinatorial clock auctions always lead to high efficiency? an experimental analysis of spectrum auction designs. *Experimental Economics* **16**(4) 511–545.

Day, R., P. Cramton. 2012. The quadratic core-selecting payment rule for combinatorial auctions. *Operations Research* **60** 588–603.

Day, Robert. 2013. The division of surplus in efficient combinatorial exchanges. *Available at SSRN: http://ssrn.com/abstract=2207067* .

Foley, D. 1967. Resource allocation and the public sector. *Yale Economic Essays* 45–98.

Georgescu-Roegen, Nicholas. 1979. Methods in economic science. *Journal of economic issues* 317–328.

Goeree, J., C. Holt. 2010. Hierarchical package bidding: A paper & pencil combinatorial auction. *Games and Economic Behavior* **70**(1) 146 – 169. doi:10.1016/j.geb.2008.02.013.

Gomory, Ralph E, William J Baumol. 1960. Integer programming and pricing. *Econometrica: Journal of the Econometric Society* 521–550.

Green, J., J.-J. Laffont. 1977. Characterization of satisfactory mechanisms for the revelation of preferences for public goods. *Econometrica* **45** 427–438.

Gul, F., E. Stacchetti. 1999. Walrasian equilibrium with gross substitutes. *Journal of Economic Theory* **87** 95–124.

Hogan, William W, Brendan J Ring. 2003. On minimum-uplift pricing for electricity markets. *Electricity Policy Group* .

Janssen, M., V. Karamychev. 2013. Gaming in combinatorial clock auctions. Tech. rep., Tinbergen Institute.

Kelso, A. S., V. P. Crawford. 1982. Job matching, coalition formation , and gross substitute. *Econometrica* **50** 1483–1504.

Lamont, Owen A, Richard H Thaler. 2003. Anomalies: The law of one price in financial markets. *The Journal of Economic Perspectives* **17**(4) 191–202.

Levin, Jonathan, Andrzej Skrzypacz. 2014. Are dynamic vickrey auctions practical?: Properties of the combinatorial clock auction. Tech. rep., National Bureau of Economic Research.

Martin, Alexander, Johannes C Müller, Sebastian Pokutta. 2014. Strict linear prices in non-convex european day-ahead electricity markets. *Optimization Methods and Software* **29**(1) 189–221.

McKenzie, Lionel W. 1959. On the existence of general equilibrium for a competitive market. *Econometrica: journal of the Econometric Society* 54–71.

Meeus, L., K. Verhaegen, R. Belmans. 2009. Block order restrictions in combinatorial electric energy auctions. *European Journal of Operational Research* **196** 1202–1206.

Milgrom, P. 2000. Putting auction theory to work: The simultaneous ascending auction. *Journal of Political Economy* **108**(21) 245–272.

Myerson, Roger B, Mark A Satterthwaite. 1983. Efficient mechanisms for bilateral trading. *Journal of Economic Theory* **29**(2) 265 – 281. doi:http://dx.doi.org/10.1016/0022-0531(83)90048-0. URL `http://www.sciencedirect.com/science/article/pii/0022053183900480`.

O'Neill, Richard P, Paul M Sotkiewicz, Benjamin F Hobbs, Michael H Rothkopf, William R Stewart. 2005. Efficient market-clearing prices in markets with nonconvexities. *European Journal of Operational Research* **1**(164) 269–285.

Scarf, Herbert E. 1990. Mathematical programming and economic theory. *Operations Research* **38**(3) 377–385.

Scarf, Herbert E. 1994. The allocation of resources in the presence of indivisibilities. *The Journal of Economic Perspectives* 111–128.

Shapley, Lloyd S, Martin Shubik. 1971. The assignment game i: The core. *International Journal of game theory* **1**(1) 111–130.

Starr, Ross M. 1969. Quasi-equilibria in markets with non-convex preferences. *Econometrica: journal of the Econometric Society* 25–38.

Van Vyve, Mathieu, et al. 2011. Linear prices for non-convex electricity markets: models and algorithms. *Université catholique de Louvain, Center for Operations Research and Econometrics, Tech. Rep* .