Duality and Equilibrium

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Abstract

We present a novel duality in vector optimization that provides an alternative characterization of Walrasian equilibrium. We demonstrate that equilibrium existence and the welfare theorems are a direct consequence of duality. By scalarizing the vector optimization problems, we further demonstrate that Walrasian equilibria are the maximizers, and roots, of a single function of allocation and prices – the economy's *potential*. We illustrate the usefulness of the potential for computing equilibria.

Keywords: *Duality, vector optimization, potential, utility clearing, Yquilibrium, Walrasian equilibrium, Pareto optimality*

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"A characterization of a model or a concept in the dual space gives usually a better insight in the problem and its solution. There are, however, also direct applications of duality theory." — P.H.M. Ruys & H.N. Weddepohl (1979)

1. Introduction

Perhaps the simplest example of duality is the equivalence between the maximal elements of a lowerset and the minimal elements of its complement. Recall that U is a lowerset in the positive orthant if, for all $u \in U$ and $x \geq 0$, $x \leq u$ implies $x \in U$. Here the partial order is the usual one, i.e. $x \leq u$ if and only if the weak inequality holds componentwise. An element $u \in U$ is maximal if $u' \geq u$ for some $u' \in U$ implies $u' = u$. In the left panel of Figure 1 the light area corresponds to the lowerset U and the thick red curve that forms its boundary shows the maximal elements of U. The darker area labeled V is the complement of U. This is an upperset whose minimal elements are also indicated by the red curve.[1] Obviously, the maximal elements of U coincide with the minimal elements of V.

Notwithstanding its simplicity, this geometric duality implies core results of general equilibrium theory. The set U is the utility possibility set, i.e. the set of utility vectors $(u_1(x_1),\ldots,u_n(x_n))$ where the bundles x_i belong to some feasible set F. The set V is the indirect utility possibility set, i.e. the set of indirect utility vectors $(v_1(p,m_1),\ldots,v_n(p,m_n))$ where the prices and incomes belong to some dual F^*. For instance, in the left panel of Figure 1, both consumers' utilities are $u(x,y) = \sqrt{x}+\sqrt{y}$ and there is one unit of both goods in the economy. If we normalize prices to sum to one, i.e. the price vector is $(p,1-p)$, then incomes also sum to one. Consumers' indirect utilities are

$$v(p,m) = \left(\sqrt{\tfrac{p}{1-p}} + \sqrt{\tfrac{1-p}{p}}\right)\sqrt{m} \tag{1}$$

and V consists of all pairs $(v(p,m), v(p,1-m))$ with p and m ranging from 0 to 1.

The frontier of U that is indicated by the red curve captures the *Paretian perspective* that the economy will maximize the utility for the collective, i.e. any gains from

[1] Recall that V is an upperset of X if, for all $v \in V$ and $x \in X$, $x \geq v$ implies $x \in V$. An element $v \in V$ is minimal if $v' \leq v$ for some $v' \in V$ implies $v' = v$.

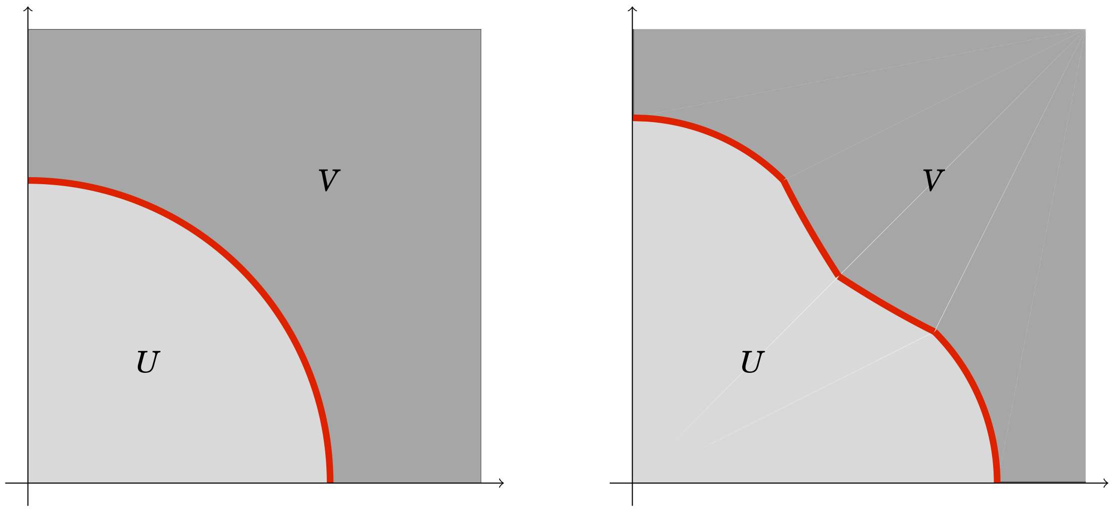

Figure 1: In the left panel, the lowerset U corresponds to the utility possibility set for a two-consumer economy with utilities $u(x, y) = \sqrt{x} + \sqrt{y}$. Its complement is the upperset V that consists of all indirect utility pairs $(v(p, m), v(p, 1-m))$ with $v(p, m)$ as in (1) and m and p between 0 and 1. The red frontier consists of maximal elements of U and minimal elements of V. Utilities in this frontier arise from price-allocation pairs that form a Walrasian equilibrium for the associated income distribution. The right panel shows that U and V also share a common frontier when U is not convex, which occurs when utilities are not concave (nor concavifiable), see Section 2.1.

trade will be seized and the final allocations are Pareto optimal.[2] The set V captures the *Walrasian perspective* that consumers maximize utility at given prices, i.e. everyone is a price taker. Feasibility dictates that, in any equilibrium of the economy, consumers' utilities belong to U. The Walrasian assumption of price-taking dictates that consumers' utilities must belong to V. Since U and V share a common frontier, **Walrasian equilibrium exists**. For any distribution of incomes there exists a price such that the affordable and optimal consumption bundles (yielding utilities in V) are also feasible (yielding utilities in U).

Moreover, **duality directly yields the Fundamental Welfare Theorems**: any Walrasian allocation is Pareto optimal and any Pareto optimal allocation is part of a Walrasian equilibrium. Finally, the duality result provides an alternative to the usual interpretation that equilibrium prices are market clearing – namely that they

[2]"The members of a collectivity enjoy maximum utility in a certain position when ... any small displacement in departing from that position necessarily has the effect of increasing the utility which certain individuals enjoy and decreasing that which others enjoy," Pareto (1906).

are **utility clearing**, i.e. the utilities consumers expect given their incomes and market prices match the utilities they receive from their allocations.

Importantly, duality establishes existence without relying on fixed-point arguments. In addition, we will show that strict quasiconvexity of the indirect utilities (as we assume throughout) ensures that the Walrasian equilibrium price is unique for a given income distribution. These results contrast with existing approaches to general equilibrium and challenge the prevailing opinion that "establishing existence should involve fixed-point arguments." The difference is that the duality result pertains to economies parameterized by income distributions rather than endowments. This alternative parametrization allows us to build on a simple duality between direct and indirect utility functions (Diewert, 1974; Crouzeix, 1983) and establish existence of a unique equilibrium price vector for any income distribution.

What does our duality result imply for economies parameterized by endowments? First, one cannot expect uniqueness of the equilibrium price, which hinges on the assumption that the $v_i(p, m_i)$ are strictly quasiconvex in prices. When incomes are functions of prices, i.e. $m_i = \langle p|\omega_i\rangle = \sum_k p_{ik}\omega_{ik}$ with ω_{ik} consumer i's endowment of good k, then the indirect utility $v_i(p, \langle p|\omega_i\rangle)$ is obviously not necessarily (strictly) quasiconvex. What about existence? Previous existence proofs entail fixed-point arguments that are complicated by the possibility that consumers' demands are unbounded when a price is zero. Our duality approach avoids this problem altogether as no (excess) demands are computed and no market clearing condition is imposed. Instead, existence of equilibrium prices follows from a geometric argument based on utility clearing.

We demonstrate that for incomes that are arbitrary functions of prices, i.e. $m_i = f_i(p)$ where $f(p) = (f_(p), \ldots, f_n(p))$ is a continuous map from the price simplex to the income simplex, there always exist a price that is utility clearing. A fortiori, Walrasian equilibrium exists for any economy parameterized by endowments as a simple example of our construction is $f_i(p) = \langle p|\omega_i\rangle$. Existence does require a fixed-point argument, unlike when economies parameterized by incomes, due to the map between prices and incomes. However, we show that Brouwer's fixed-point theorem suffices unlike Arrow and Debreu's (1954) proof that requires Kakutani's theorem.

Besides streamlining the standard textbook results of existence and the welfare theorems, we put our duality result to work by operationalizing the notion of utility clearing. To this end we introduce the **economy's potential**, which is a single function of allocations and prices:

$$Y_\alpha(x,p;\omega) = \sum_i \alpha_i(u_i(x_i) - v_i(p, \langle p|\omega_i\rangle)) \tag{2}$$

The program we consider is to maximize the potential (2) with respect to prices and allocations, subject to budget constraints. Since the potential is a continuous function over a compact domain it has one or more maxima, which we term **Yquilibria**. We demonstrate that the set of Yquilibria coincides with the set of Walrasian equilibria of the economy. We further show that they are roots of the potential, which reflects the utility clearing nature of equilibrium prices.

Obtaining market outcomes via an optimization program provides a powerful computational approach to market design. In many applications, non-economic constraints such as legal and political constraints or fairness and complexity considerations play a role. These non-economic factors supplement the usual feasibility and budget constraints and are readily incorporated into an optimization approach, but might be hard to deal with otherwise. A recent example of this computational approach is (Bichler et al., 2018, 2019) who built a combinatorial exchange for trading catch shares in New South Wales (NSW), Australia. The optimization program entailed a hierarchy of objectives, the top one being economic efficiency, that were introduced to ensure participation by all stakeholders.[3]

Besides its use for market design, the potential improves on existing approaches to compute Walrasian equilibria. A well-known method due to Negishi (1960) finds Pareto optimal allocations by maximizing a linear welfare function over a strictly convex utility possibility set (and then solves for prices using the budget constraints).

[3]The exchange ended two decades of political debate by providing a market-based response to a major policy problem faced by fisheries worldwide: the reallocation of catch shares in cap-and-trade programs designed to prevent overfishing. The design question was how to effectively reallocate shares using a market that employed linear and anonymous prices, which were deemed necessary for reasons of simplicity, transparency, and fairness. The exchange matched 86% of active fishers' bids and reduced their share deficits by 95% in high-priority classes. In addition, 62 businesses successfully exited the industry by selling all their shares, receiving \$10.1 million in total. The NSW government had set aside \$15 million to subsidize the market, but spent only \$11.6 million because higher subsidy levels did not significantly raise their objectives. This saved NSW taxpayers \$3.4 million.

Strict convexity of the utility possibility set is crucial for Negishi's (1960) method. A recent contribution by Che et al. (2024) shows how to find Pareto optimal allocations for a utility possibility set that is *convex but not strictly convex*, i.e. its boundary may have "flat" parts that correspond to weakly Pareto optimal allocations.

Our duality result allows for a much more general approach that applies to utility possibility sets that are *not convex at all*. The right panel of Figure 1 shows the utility possibility set for the economy in Section 2.1. As in the convex case of the left panel, its complement is the indirect utility set and the two sets share a common frontier. This frontier consists of direct utilities that arise from Pareto optimal allocations and of indirect utilities that arise from Walrasian equilibrium prices. These allocations and prices can be readily obtained by maximizing the potential. In contrast, Negishi's (1960) method cannot be applied nor can the extension by Che et al. (2024).

1.1. Organization

Section 2 provides a novel duality result in vector optimization that offers an alternative interpretation of general equilibrium and paves the way for an optimization approach to computing equilibrium. Section 3 introduces the economy's potential, details the optimization program, and compares it to prior approaches. Section 4 concludes and discusses future directions. Proofs can be found in the Appendix.

2. Duality in Vector Optimization

Consider an exchange economy with $\mathscr{N} = \{1,\dots,N\}$ consumers and $\mathscr{K} = \{1,\dots,K\}$ goods. For $k \in \mathscr{K}$, let $W_k > 0$ denote the total amount of good k. The set of feasible allocations is

$$F(W) = \{x \in \mathbb{R}^{NK}_{\geq 0} \mid \sum_{i \in \mathscr{N}} x_{ik} \leq W_k \;\; \forall\, k \in \mathscr{K}\}$$

For vectors $v, v' \in \mathbb{R}^K$ let $\langle v | v' \rangle = \sum_{k \in \mathscr{K}} v_k v'_k$ denote the usual inner product. Without loss of generality, we normalize prices such that the economy's total income is one. The set of prices is then an asymmetric simplex, $\Sigma_K(W) = \{p \in \mathbb{R}^K_{\geq 0} \mid \langle p | W \rangle = 1\}$, and the set of income distributions $m = (m_1, \dots, m_N)$ is Σ_N. For $i \in \mathscr{N}$, consumer i's budget set is

$$B_i(p, m_i) = \{x_i \in \mathbb{R}^K_{\geq 0} \,|\, \langle p|x_i\rangle \leq m_i\}$$

and consumer i's utility function $u_i : \mathbb{R}^K_{\geq 0} \to \mathbb{R}$ is assumed to be increasing, quasiconcave, and differentiable. Consumer i's bundle x_i is affordable if $x_i \in B_i(p, m_i)$ and optimal if $u_i(x_i) \geq u_i(x'_i)$ for all $x'_i \in B_i(p, m_i)$. The concatenation of the x_i for $i \in \mathcal{N}$ defines an allocation x.

Definition 1 *A Walrasian equilibrium consists of a price $p \in \Sigma_K(W)$ and a feasible allocation $x \in F(W)$ such that, for $i \in \mathcal{N}$, the x_i are affordable and optimal.*

Consumer i's indirect utility function

$$v_i(p, m_i) = \max_{\langle p|x_i\rangle = m_i} u_i(x_i)$$

has the following properties.

Lemma 1 *The indirect utility $v_i(p, m_i)$ is*

(i) homogeneous of degree zero in income and prices,

(ii) non-increasing in prices and strictly increasing in income,

(iii) continuous and strictly quasiconvex (but not necessarily differentiable).

(iv) The dual of $v_i(p, m_i)$ is the utility function, i.e. $u_i(x_i) = \min_{\langle p|x_i\rangle = m_i} v_i(p, m_i)$.

Properties (i)-(iii) are standard, e.g. Mas Colell et al. (1995, Prop. 3.D.3). We assume utilities are differentiable, which holds if and only if indirect utility is *strictly* quasiconvex so that it has a unique minimizer (Crouzeix, 1983). Property (iv) is due to Diewert (1974) and Crouzeix (1983) who use homogeneity of degree zero to normalize income to one. This is natural when studying a single consumer. However, when studying the entire economy it will prove useful to consider different income distributions.

The utility possibility set is defined as[4]

$$\text{UPS} = \left\{(u_1(x_1), \ldots, u_N(x_N)) \,|\, x \in F(W)\right\}$$

[4]This definition differs from the usual one that adds the negative orthant to every element of the UPS, see e.g. Mas Colell et al. (1995, p. 818). The two definitions coincide when $u_i(0) = -\infty$ for $i \in \mathcal{N}$, e.g. when $u_i(x_i) = \sum_{k \in \mathcal{K}} a_{ik} \log(x_{ik})$ for some non-negative a_{ik}.

For $u, u' \in \mathbb{R}^N$ let $u' \geq u$ mean that $u'_i \geq u_i$ for $i \in \mathcal{N}$. A **maximal element** of the UPS is a vector of utilities $u \in$ UPS such that $u' \geq u$ for some $u' \in$ UPS implies $u' = u$. We also define the **indirect utility possibility set**, which is novel to the literature:

$$\text{VPS} = \left\{(v_1(p,m_1),\ldots,v_N(p,m_N)) \,|\, (p,m) \in F^*(W) = \Sigma_K(W) \times \Sigma_N\right\}$$

where the incomes m_i are non-negative and sum to one, i.e. $m = (m_1,\ldots,m_N) \in \Sigma_N$. A **minimal element** of the VPS is a vector of indirect utilities $v \in$ VPS such that $v' \leq v$ for some $v' \in$ VPS implies $v' = v$.

An optimal **solution** to the maximization problem $\max_{x \in F(W)}(u_1(x_1),\ldots,u_N(x_N))$ is a feasible allocation $x \in F(W)$ such that $(u_1(x_1),\ldots,u_N(x_N))$ is a maximal element of the UPS, i.e. it belongs to the UPS' upper frontier. Likewise, an optimal solution to the minimization problem $\min_{(p,m)\in F^*(W)}(v_1(p,m_1),\ldots,v_N(p,m_N))$ is a price $p \in \Sigma_K(W)$ and an income distribution $m \in \Sigma_N$ such that $(v_1(p,m_1),\ldots,v_N(p,m_N))$ is a minimal element of the VPS, i.e. it belongs to the VPS' lower frontier. The next theorem shows that the UPS and VPS intersect only along their frontiers.

Theorem 1 *u is a maximal element of the* UPS *iff it is a minimal element of the* VPS:

$$\max_{x \in F(W)} (u_1(x_1),\ldots,u_N(x_N)) = \min_{(p,m)\in F^*(W)} (v_1(p,m_1),\ldots,v_N(p,m_N)) \tag{3}$$

Moreover, for any $m \in \Sigma_N$ there is a unique solution to the right side of (3)*, which is the Walrasian equilibrium price for the economy with income distribution m.*

Remark 1 *Existence of Walrasian equilibrium prices follows from duality and does not require any fixed-point arguments, see the Appendix for details.*

The proof of (3) is based on two lemmas, see the Appendix. The first lemma shows that any element of the intersection UPS ∩ VPS must be a maximal element of the UPS and a minimal element of the VPS. In other words, the intersection UPS ∩ VPS does not contain interior elements of the UPS nor of the VPS. The second lemma shows that maximal elements of the UPS belong to the VPS and minimal elements of the VPS belong to the UPS, i.e. their frontiers *are* in the intersection. Proof of existence follows from a standard argument in vector optimization and uniqueness follows from strict quasiconcavity of the indirect utilities, see the Appendix.

2.1. Fenchel's Economy

In this section we consider a simple economy with two goods and two consumers who have quasiconcave utilities that are not concavifiable. This economy illustrates key features of our duality result. First, the duality result of Theorem 1 applies even when the UPS is **non-convex**. This is surprising given the deep connection between duality and convexity. *Strong duality*, i.e. the absence of a "duality gap," typically only holds for convex optimization problems.[5] Second, the duality result applies even when the Pareto-optimal allocations lie on the boundary of the Edgeworth box and the standard condition of equal marginal-rates-of-substitution does not apply. Third, while the Walrasian equilibrium price is unique for any income distribution, there may be multiple allocations that yield utilities that match the indirect utilities at this price.

The utility function of both consumers is given by

$$u(x,y) = x + \sqrt{y + x^2} \tag{4}$$

The lighter area in the left panel of Figure 2 shows the UPS, which is not convex nor can it be convexified by "concavifying" the utility function in (4).[6] Moreover, the usual approach of equating marginal rates of substitution to the price ratio does not apply as the optimal allocations are not necessarily interior. Instead, we derive the optimal allocations by applying Roy's identity to the indirect utility

$$v(p,q,m) = \max\Big(\frac{2m}{p}, \sqrt{\frac{m}{q}}\Big)$$

The darker area in the left panel of Figure 2 shows the VPS. The red curves show the indirect utility pairs $(v(p,1-p,m), v(p,1-p,1-m))$ for different levels of m as a function of $p \in (0,1)$. The VPS is the union of all such curves. For each m there is a unique price such that the indirect utility pair belongs to the (lower) frontier of the

[5]Even convex optimization problems typically require additional constraint qualification conditions (such as Slater's condition) for strong duality to apply, see e.g. (Boyd and Vandenberghe, 2004, p.226).

[6]The reason is that their indifference curves are non-parallel lines. It can be shown this implies there is no monotonic transformation of the utility such that it becomes concave (Reny, 2013), an observation originally due to Fenchel.

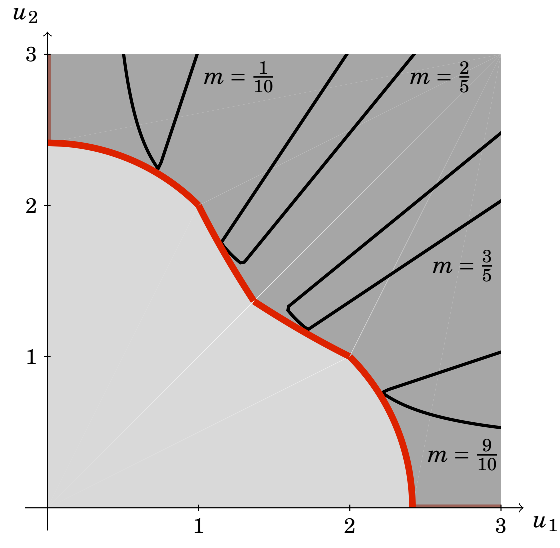

Figure 2: In the left panel, the lighter area corresponds to the UPS and the darker area to the VPS for $u_1(x,y) = u_2(x,y) = x + (y + x^2)^{1/2}$. The sets overlap only along their frontiers indicated by the red curve. The black curves show the indirect utility pairs $(v(p,1-p,m), v(p,1-p,1-m))$ for various income levels as functions of p. In the right panel, the red polyline shows Pareto optimal allocations in the Edgeworth box and the blue lines show endowments resulting in the same Pareto optimal allocation.

VPS and the (upper) frontier of the UPS, indicated by the red curve.

The indirect utility is differentiable when the arguments of the max differ in which case Roy's identity produces an optimal demand for only one of the goods. The resulting Walrasian equilibria lie on the boundary of the Edgeworth box, i.e. one consumer receives only one of the goods while the other is indifferent.[7] Interior optimal allocations require $2m/p = \sqrt{m/(1-p)}$ and $2(1-m)/p = \sqrt{(1-m)/(1-p)}$, which implies $m = \frac{1}{2}$ and $p = \sqrt{3} - 1$. Consumer 1's optimal allocation (x_{11}, x_{12}) then lies on the line

$$x_{12} = 1 + \tfrac{1}{2}\sqrt{3} - (1 + \sqrt{3})x_{11}$$

for $\frac{1}{4}(3 - \sqrt{3}) \le x_{11} \le \frac{1}{4}(1 + \sqrt{3})$. The thick red polyline in the right panel of Figure 2 shows all Pareto-optimal allocations in the Edgeworth box. The four segments on the edges of the Edgeworth box correspond to the four different parts of the UPS' frontier in the left panel. The line that crosses from the top to the bottom part of the

[7] For instance, for $0 \le m \le \frac{1}{5}$, consumer 1 receives nothing of good 1 and $m/(1-p)$ of good 2 where the price p is such that consumer 2 is indifferent, i.e. $p = 2\sqrt{(2-m)(1-m)} - 2(1-m)$.

Edgeworth box corresponds to a single point in the left panel, i.e. the "kink" on the 45-degree line. This line illustrates that for a given income distribution, equilibrium allocations need not necessarily be unique (but the equilibrium price is). The line coincides with both consumers' indifference curves and both their utilities are $u_1 = u_2 = \frac{1}{2}(1+\sqrt{3})$ for any allocation on the line.

The blue lines in the right panel of Figure 2 are the budget lines for the equilibrium price and indicate endowments resulting in the same optimal allocation. Note that the same allocation can occur for different incomes, e.g. when $\frac{1}{5} \le m \le \frac{1}{3}$ several blue lines end up in the upper-left corner where $x_1 = (0,1)$ and $x_2 = (1,0)$. However, their slopes differ, reflecting a different equilibrium price.

2.2. Welfare Theorems

The solutions to the left side of (3) are Pareto optimal allocations that yield utilities in the UPS' frontier. The solutions to the right side are incomes m_i and a price vector p that yield utilities $v_i(p,m_i) = u_i(x_i(p,m_i))$ that belong to the same frontier. Hence, the optimal demands are feasible and (x,p) with $x = (x_1(p,m_1),\ldots,x_N(p,m_N))$ is the Walrasian equilibrium for the economy with incomes $(m_1,\ldots,m_N)$.

Corollary 1 *The duality result* (3) *encapsulates the first and second welfare theorem: Walrasian equilibrium allocations are Pareto optimal and any Pareto optimal allocation is part of a Walrasian equilibrium.*

Uniqueness of the equilibrium price in Theorem 1 requires strict quasiconvexity of the indirect utilities so that they have a single minimizer. Strict quasiconvexity of the indirect utility holds if and only if the utility function is differentiable. If this assumption is relaxed then there can be multiple Walrasian equilibria. Consider, for instance, an economy with Leontief utilities: $u_i(x,y) = \min(x,y)$ for $i = 1,2$. The indirect utilities are $v_i = m_i/(p+q)$, which have linear iso-indirect-utility curves. The Walrasian equilibria for this Leontief economy are $x_1 = (m,m)$ and $x_2 = (1-m,1-m)$ for $m \in [0,1]$ and *any* prices p and q that sum to one.

2.3. Endowments, Fixed Points, and Equilibrium Multiplicity

When the economy is parameterized by endowments rather than incomes, the equilibrium price need not be unique even if utilities are differentiable. The reason is simple. For fixed m_i, the indirect utility $v_i(p, m_i)$ is (assumed to be) strictly quasiconcave in prices. However, when incomes vary with prices, i.e. $m_i = \langle p|\omega_i\rangle$ where ω_i denotes i's endowment, the indirect utility $v_i(p, \langle p|\omega_i\rangle)$ is not necessarily (strictly) quasiconcave in prices. Hence, the equilibrium price cannot be expected to be unique.

A related issue is equilibrium existence. Previous existence proofs entail fixed-point arguments that are complicated by the possibility that consumers' excess demands are unbounded if a price is zero. We completely avoid this issue by not computing any excess demand nor imposing market clearing conditions. Instead, we ask if there are prices such that indirect utilities match the utilities of feasible allocations, i.e. whether there exist utility clearing prices. This is essentially a geometric exercise: do the black curves of Figure 2 still intersect the red frontier once the incomes m_i in $v_i(p, m_i)$ depend on prices?

We demonstrate they do even when incomes are *general* functions of prices. To this end, consider the parameterized $(K-1)$-dimensional surface

$$S_f = \{(v_1(p, f_1(p)), \ldots, v_N(p, f_N(p))) \,|\, p \in \Sigma_K(W)\}$$

where $f = (f_1, \ldots, f_N)$ is some continuous function that maps price vectors to income distributions. By construction, $S_f \subset \text{VPS}$. The question is whether S_f intersects the UPS. We will demonstrate it does for any continuous function $f : \Sigma_K(W) \to \Sigma_N$.

Figure 3 illustrates our claim for the Fenchel economy of Section 2.1. The lighter areas show the same non-convex UPS as in Figure 2 and the darker areas show the same complementary VPS. The thick red curves show their common frontiers. The black curves correspond to indirect utility pairs $(v(p, 1-p, f(p)), v(p, 1-p, 1-f(p)))$ where $f(p) = \alpha|\cos(1/(2p(1-p)))|$ with $\alpha = \frac{3}{5}$ in the left panel and $\alpha = \frac{4}{5}$ in the right panel. In the former case, there is a unique intersection of the black curve with the red frontier, in the latter case there are two.

Why is there at least one intersection of the indirect utility curves with the UPS? Again the reason is simple. The Walrasian price correspondence is upper-hemi-continuous, see e.g. Hildenbrand and Mertens (1972). By Theorem 1 this correspondence is single valued when parameterized by income distributions. Hence, the map $P : \Sigma_N \to \Sigma_K(W)$, which assigns Walrasian equilibrium prices to income distributions,

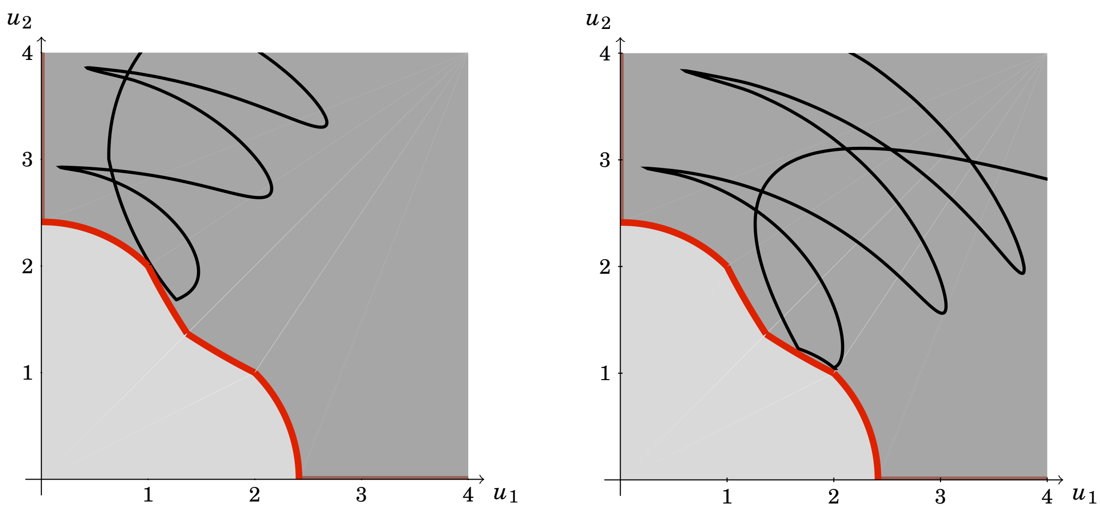

Figure 3: The UPS (light) and VPS (dark) are the same as in Figure 2. The black curves correspond to indirect utility pairs $(v(p,1-p,f(p)), v(p,1-p,1-f(p)))$ where $f(p) = \alpha|\cos(1/(2p(1-p)))|$ with $\alpha = 3/5$ on the left and there is a unique intersection with the red curve and $\alpha = 4/5$ on the right and there are two intersections.

is both upper-hemi-continuous and single valued, i.e. it is a continuous function. By Brouwer's fixed-point theorem, $f \circ P : \Sigma_N \to \Sigma_N$ thus has a fixed-point.

Let m denote a fixed-point of $f \circ P$ and let $p = P(m)$ then

$$S_f \ni \big(v_1(p, f_1(p)), \ldots, v_N(p, f_N(p))\big) \in \text{UPS} \cap \text{VPS}$$

i.e. S_f intersects the UPS. A simple example of this construction is $f_i(p) = \langle p|\omega_i \rangle$, which satisfies $f_i(p) \geq 0$ and $\sum_{i \in \mathcal{N}} f_i(p) = \langle p|W \rangle = 1$. Hence, Walrasian equilibria exist for any choice of endowments and yield utilities in UPS ∩ VPS. Since elements in this intersection are uniquely characterized by some $(m_1, \ldots, m_N) \in \Sigma_N$, each can be recovered by choosing $\omega_i = m_i W$ so that $f_i(p) = m_i$ for $i \in \mathcal{N}$. This can be accomplished via redistribution, i.e. consumers receive additional endowments $m_i W - \omega_i$ for $i \in \mathcal{N}$. Note that $\sum_{i \in \mathcal{N}} m_i W - \omega_i = 0$ so this redistribution is balanced.

Corollary 2 *For an economy parameterized by endowments, one or more Walrasian equilibria exist and always belong to* UPS ∩ VPS. *Hence, any Walrasian equilibrium allocation is Pareto optimal (first welfare theorem). All Pareto optimal allocations can be obtained as part of a Walrasian equilibrium by choosing "diagonal" endowments $\omega_i = m_i W$ for $i \in \mathcal{N}$ and varying $m = (m_1, \ldots, m_N)$ over Σ_N (second welfare theorem).*

3. The Economy's Potential

We operationalize the duality result of Theorem 1 by scalarizing the vector optimization programs in (3). For $\alpha_i > 0$, let $U_\alpha(x) = \sum_i \alpha_i u_i(x_i)$ denote social welfare and $V_\alpha(p;\omega) = \sum_i \alpha_i v_i(p, \langle p|\omega_i\rangle)$ its dual. Their difference is the economy's potential.

Definition 2 *For $\alpha \in \Sigma_N$ the economy's **potential**, $Y_\alpha : F(W) \times \Sigma_K(W) \to \mathbb{R}_{\leq 0}$, is*

$$Y_\alpha(x,p;\omega) = \sum_{i \in \mathcal{N}} \alpha_i \big(u_i(x_i) - v_i(p, \langle p|\omega_i\rangle) \big) \tag{5}$$

The potential is continuous and non-positive everywhere on its compact domain.

Definition 3 *A **Yquilibrium** is an allocation-price pair (x,p) that solves*

$$\max_{\substack{x \in F(W),\, p \in \Sigma_K(W) \\ \langle p|x_i\rangle = \langle p|\omega_i\rangle}} Y_\alpha(x,p;\omega) \tag{6}$$

Even though x only enters the utility part and p only enters the indirect utility part, the budget constraint precludes the maximization problem to be split in two.

The potential is a continuous function over a compact domain, so Bolzano's theorem ensures that it must attain a maximum at least once. In other words, there is a solution to (6) and a Yquilibrium exists. Since the endowments ω_i enter the objective and constraints in (6) only in the form of incomes $\langle p|\omega_i\rangle$, this solution is the same for all endowments that yield the same incomes at the equilibrium price. These endowments lie on a plane, see e.g. the blue lines in the right panel of Figure 2.

Theorem 2 *The set of Yquilbria is identical to the set of roots of the potential and is independent of the welfare weights. The set of Yquilibria coincides with the set of Walrasian equilibria.*

Independence of the welfare weights stems from the fact that each consumer's utility clears in a Walrasian equilibrium, i.e. $u_i(x_i) = v_i(p, \langle p|\omega_i\rangle)$ for $i \in \mathcal{N}$, so any linear combination of their difference vanishes. We introduce them nonetheless to be comparable to prior approaches, see the next section, and because they can be useful instruments in market design applications, see Section 4.

3.1. Prior Approaches

Negishi's (1960) method determines Pareto-optimal allocations as functions of the weights α_i by maximizing social welfare U_α. The correct weights then follow from a system of fixed-point conditions that ensure consumers' budget constraints are met.[8] Concave utilities are required for Negishi's method so that the UPS is strictly convex. Section 2.1 provides an example where the UPS is not convex nor can it be convexified by concavifying the utilities. In such cases, Negishi's method fails.[9]

For an auction context, Ausubel (2006) derives equilibrium prices by minimizing a Lyapunov function that can be shown to be equal to dual social welfare.[10] Like Negishi, Ausubel requires concave utilities that are further assumed to be quasilinear. This latter constraint is particularly restrictive in that it rules out any income effects. Without quasilinearity, existence of a Lyapunov function that ensures price convergence cannot be guaranteed.[11]

Remark 2 *Compared to prior approaches that are curtailed to restricted domains and rely on fixed-point methods, the program in* (6) *offers the following advantages:*

- *It allows for income effects and does not require quasilinearity.*
- *It works for any UPS and does not require concave (or concavified) utilities.*
- *It determines allocations and prices that are independent of the* α_i *in* (5).
- *It dispenses with the need for solving a system of fixed-point equations.*

[8]Negishi (1960) shows that the correct weights are equal to the inverse of the marginal utilities of income, i.e. $\alpha_i = 1/(\partial v_i/\partial m_i)$ for $i \in \mathcal{N}$. These conditions define a system of fixed-point equations: the inverse marginal utilities of income on the right depend on prices, which in turn depend on the α_i weights on the left. Intuitively, these weights are such that the impact on welfare of an extra dollar to the economy is independent of who receives it.

[9]While the UPS in the left panel of Figure 2 cannot be convexified by concavifying the utility functions, its convex hull can be obtained in a "mechanical fashion" by connecting the two off-diagonal kink points via a line. Negishi's method fails in that it can only produce points in the frontier of this convex hull. But not, for instance, the kink point on the 45-degree line, which corresponds to the interior Pareto optimal allocations shown in the right panel of Figure 2.

[10]Ausubel (2006) defines the Lyapunov function $L(p) = \langle p|W\rangle + \sum_i v_i(p)$ where $\langle\cdot|\cdot\rangle$ denotes the inner product, W_k is the total amount of good k, and $v_i(p) = \max_{x_i}(u_i(x_i) - \langle p|x_i\rangle)$ is the Fenchel dual of $u_i(x_i)$. Since $W_k = \sum_i \omega_{ik}$, with ω_{ik} consumer i's endowment of good k, it is straightforward to rewrite the Lyapunov function as $L(p) = \sum_i v_i(p, \langle p|\omega_i\rangle)$ where $v_i(p, \langle p|\omega_i\rangle) = \max_{x_i}(u_i(x_i) + \langle p|\omega_i\rangle - \langle p|x_i\rangle)$. This Lyapunov function equals dual social welfare $V_\alpha(p;\omega) = \sum_i \alpha_i v_i(p, \langle p|\omega_i\rangle)$ when $\alpha_i = 1$.

[11]Scarf (1960), for instance, considers a Leontief economy with income effects and finds that prices cycle forever.

3.2. Potential Maximization

An easy demonstration of potential maximization concerns the economy of the Introduction which has two consumers with utilities $u(x,y)=\sqrt{x}+\sqrt{y}$. If we set welfare weights to one and use feasibility the potential can be written as

$$Y = \phi(x_{11})+\phi(x_{12})-\phi(m)\left(\sqrt{\tfrac{p}{1-p}}+\sqrt{\tfrac{1-p}{p}}\right)$$

where $\phi(x)=\sqrt{x}+\sqrt{1-x}$, consumers' allocations are (x_{11},x_{12}) and $(1-x_{11},1-x_{12})$ respectively, and $m = p\omega_{11}+(1-p)\omega_{12}$. The potential's maximizers are its roots, which are easy to find: $x_{11}=x_{12}=m$ and $p=\frac{1}{2}$. This yields utilities that satisfy $u_1^2+u_2^2=4$ and that produce the UPS' frontier in the left panel of Figure 1.

Thus far we assumed that utility functions are increasing, which means that the UPS' frontier has no "flat" parts. For one consumer's utility to be raised, the utility of one or more others will have to be lowered. This is not necessarily the case when the UPS' frontier has flat parts, i.e. when the UPS is *convex but not strictly convex* and there are weakly Pareto optimal allocations.[12] This can occur when utilities are non-decreasing instead of increasing.

To illustrate, consider an economy with two goods, one unit of each, and two consumers with utilities

$$\begin{aligned} u_1(x,y) &= \log(\min(x,2y)) \\ u_2(x,y) &= \log(x+y) \end{aligned} \tag{7}$$

The additional log ensures the UPS is a lowerset in $\mathbb{R}^2$ rather than its positive orthant.[13] Obviously, the utilities represent the same preferences with or without the log. The light area in the left panel of Figure 4 shows the UPS and the dark area the VPS. The latter consists, as usual, of all pairs $(v_1(p,m),v_2(p,1-m))$ where p and m

[12] A feasible allocation is weakly Pareto optimal if there is no other feasible allocation that makes everyone better off.

[13] When a UPS with flat parts is a lowerset in the positive orthant, its complement can have artificial minimal elements at one of the axes.

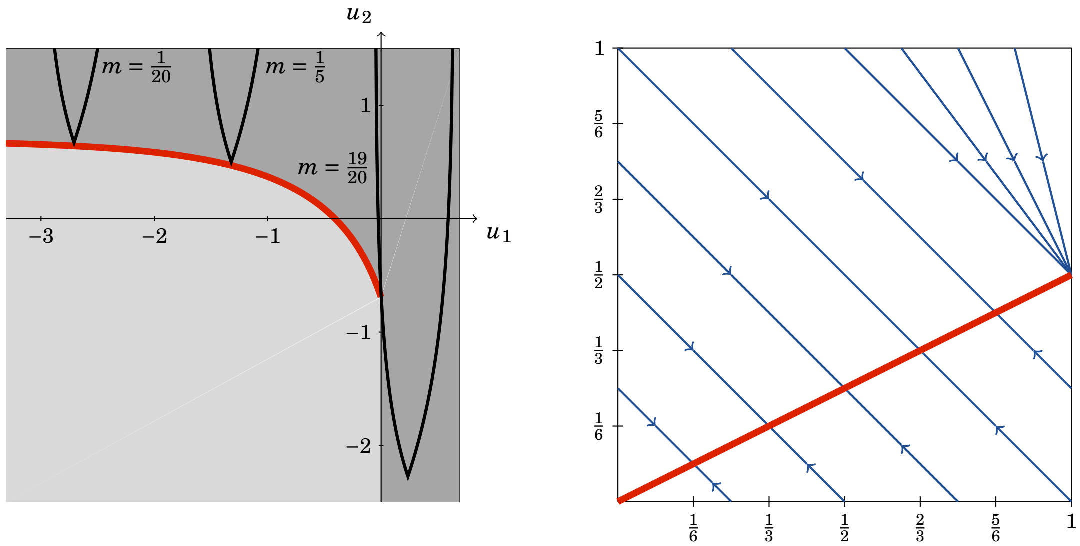

Figure 4: In the left panel, the lighter area corresponds to the UPS and the darker area to the VPS for the utilities in (7). The red curve shows maximal elements of the UPS and minimal elements of the VPS. The black curves correspond to the indirect utility pairs $(v_1(p,m), v_2(p,1-m))$ for various income levels as functions of p. In the right panel, the red line shows Pareto optimal allocations in the Edgeworth box and the blue lines show endowments resulting in the same Pareto optimal allocation.

range from 0 to 1 and the indirect utilities are

$$\begin{aligned} v_1(p,m) &= \log\Big(\frac{2m}{1+p}\Big) \\ v_2(p,m) &= \log\Big(\max\Big(\frac{1-m}{p}, \frac{1-m}{1-p}\Big)\Big) \end{aligned} \tag{8}$$

The thick red curve shows the maximal elements of the UPS and the minimal elements of the VPS. While the minimal and maximal elements are the same, they do not span the entire frontier. In particular, utility pairs on the $u_1 = 0$ axis that correspond to weakly Pareto optimal allocations are *not* part of the red curve.

A recent contribution by Che et al. (2024) shows how to refine Negishi's approach to find only those Pareto optimal allocations that yield utilities on the red curve. We derive these allocations via potential maximization. The latter has the advantage that it also produces equilibrium prices, without solving fixed-points, see Remark 2.

Potential maximization is straightforward and can be done using the graphical tools developed above. The black "V" shaped curves in the left panel of Figure 4 show indirect utility pairs $(v_1(p,m), v_2(p,1-m))$ as functions of the price for three income levels. Since the potential's maxima are roots, we look for allocations and prices for which these curves touch the UPS. For $m \le \frac{3}{4}$ this occurs when v_2 is minimized at $p = \frac{1}{2}$. Consumer 1's allocation is $(\frac{4}{3}m, \frac{2}{3}m)$ and consumer 2 gets the remainder. For $m > \frac{3}{4}$ the black curve touches the UPS but at a higher price. Consumer 1's allocation is $(1, \frac{1}{2})$ so the equilibrium price is $p = 2m - 1$. The right panel of Figure 4 shows the same results in the Edgeworth box. The red line corresponds to the Pareto optimal allocations and the blue lines indicate endowments that yield the same allocation.

More generally, let $m \in \Sigma_N^{int}$ denote any strictly positive income distribution, i.e. such that $m_i > 0$ for all $i \in \mathcal{N}$, and let

$$Y(x,p;m) = \sum_{i \in \mathcal{N}} u_i(x_i) - v_i(p, m_i) \tag{9}$$

denote the potential as a function of incomes (rather than endowments) where we set all welfare weights to one.

Corollary 3 *Suppose consumers' utilities are continuous and non-decreasing. An allocation x is Pareto optimal and the price $p \in \Sigma_K$ is Walrasian iff*

$$(x,p) \in \text{argmax } Y(x,p;m) \tag{10}$$

for some $m \in \Sigma_N^{int}$.

This follows from Theorem 2 and Corollary 1. The allocation-price pair for a degenerate income distribution m, which has one or more $m_i = 0$, is obtained by taking the limit $m(\varepsilon) \to m$ where the path $m(\epsilon)$ for $\varepsilon \in [0,1]$ lies entirely in Σ_N^{int}.

As in Negishi (1960) and Che et al. (2024), Pareto optimal allocations are found via maximization, but they are parameterized by incomes m_i rather than welfare weights α_i. Negishi's method requires a strictly convex UPS and Che et al.'s method a convex but not strictly convex UPS. Potential maximization can handle both these cases, see Figures 1 and 4, and even applies when the UPS is not convex at all, see Figure 2. Moreover, potential maximization automatically produces the associated equilibrium prices.

4. Conclusions and Outlook

We present a novel duality result in quasiconvex/quasiconcave optimization based on the notion of *utility clearing*. Maximal elements of the utility possibility set coincide with minimal elements of the indirect utility possibility set, a concept introduced in this paper. We demonstrate that this simple duality streamlines the presentation of core results in general equilibrium theory – equilibrium existence and the welfare theorems. We further show our duality result applies even if the utility possibility set is not convex, which is unexpected given the deep connection between duality and convexity. Our duality result is, to our knowledge, the most general to date.

We scalarize the vector optimization programs to obtain a standard maximization problem with a single objective – the economy's *potential*, which is a continuous function of allocations and prices that is everywhere non-positive over its compact domain. We show Walrasian equilibria are maximizers, and roots, of the potential. The potential thus offers a litmus test for equilibrium existence. Given preferences and endowments it is a mechanical exercise to compute the potential's maximum value. A Walrasian equilibrium exists if and only if this exercise returns nil.

In the presence of non-convexities it may not, as Walrasian equilibria need not exist. In this case, general equilibrium theory is quiet about the allocations and prices that ensue. But even when the potential does not have roots it has maxima, which we term Yquilibria. They are natural candidate outcomes for non-convex economies as they entail optimal allocations subject to linear and anonymous prices that are approximately utility clearing, i.e. they minimize the gap between expected utilities based on prices and realized utilities based on allocations.

We plan to explore non-convex markets through the lens of Yquilibria and test a combinatorial exchange ("YCE") that implements them (Goeree et al., 2024). The exchange allows for "all-or-nothing" combinatorial orders to protect traders from exposure problems that arise when each good is traded separately. Linear anonymous prices follow from an optimization program like (5). The weights α_i in (5) provide engineering opportunities for the designer to favor participants that submit small orders over those with large combinatorial offers. The empirical validation of YCE, and its implementation in real-world applications, is left for future research.

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A. Proofs

Proof of Theorem 1. The proof of (3) is based on two lemmas.

Lemma A1 *If $u \in UPS \cap VPS$ then u is a maximal element of the UPS and a minimal element of the VPS.*

Proof. If $u = (u_1, \dots, u_N) \in \text{UPS} \cap \text{VPS}$ there exist $x \in F(W)$ and $(p, m) \in \Sigma_K(W) \times \Sigma_N$ such that $u_i = u_i(x_i)$ and $u_i = v_i(p, m_i)$ for $i \in \mathcal{N}$. Suppose, in contradiction, that u is not a minimal element of the VPS. Then there exist $(p', m') \in \Sigma_K(W) \times \Sigma_N$ such that $u_i(x_i) \geq v_i(p', m'_i)$ for $i \in \mathcal{N}$ with strict inequality for at least one $i \in \mathcal{N}$. But $v_i(p', m'_i) = u_i(x_i(p', m'_i))$ with $x_i(p', m'_i)$ consumer i's optimal (Marshallian) demand that satisfies $\langle p' | x_i(p', m'_i) \rangle = m'_i$. Since indirect utility is strictly increasing in income (Lemma 1), we must have $\langle p' | x_i \rangle \geq m'_i$ for $i \in \mathcal{N}$ with strict inequality for at least one $i \in \mathcal{N}$, so $\sum_{i \in \mathcal{N}} \langle p' | x_i \rangle > \sum_{i \in \mathcal{N}} m'_i = 1$. However, $x \in F(W)$ implies $\sum_{i \in \mathcal{N}} \langle p' | x_i \rangle \leq \langle p' | W \rangle = 1$, which yields the desired contradiction. Likewise, suppose, in contradiction, that u is not a maximal element of the UPS. Then there exist $x' \in F(W)$ such that $u_i(x'_i) \geq v_i(p, m_i)$ for $i \in \mathcal{N}$ with strict inequality for at least one $i \in \mathcal{N}$. Since indirect utility is strictly increasing in income, we must have $\langle p | x'_i \rangle \geq m_i$ for $i \in \mathcal{N}$ with strict inequality for at least one $i \in \mathcal{N}$, so $\sum_{i \in \mathcal{N}} \langle p | x'_i \rangle > \sum_{i \in \mathcal{N}} m_i = 1$. However, $x' \in F(W)$ implies $\sum_{i \in \mathcal{N}} \langle p | x'_i \rangle \leq \langle p | W \rangle = 1$, which yields the desired contradiction. ∎

Lemma A2 *Minimal elements of the VPS belong to the UPS and maximal elements of the UPS belong to the VPS.*

The proof of Lemma A2 is based on Kuhn-Tucker conditions for vector optimization problems. These conditions are necessary (see Theorem 7.8 in Jahn, 2011) and sufficient because the vector objective consists of quasiconvex / quasiconcave functions (see Theorem 7.15 in Jahn, 2011).

Proof. Minimal elements of the VPS can be obtained from the program

$$\min_{\substack{(p,m) \in \Sigma_K(W) \times \Sigma_N \\ v_i(p, m_i) \leq \overline{v}_i \; \forall i < N}} v_N(p, m_N)$$

The first-order conditions with respect to prices and incomes are

$$\begin{aligned}\nabla_p v_N + \sum_{i<N} \lambda_i \nabla_p v_i &= \nu \quad \forall k \in \mathcal{K} \\ -\lambda_i \frac{\partial v_i}{\partial m_i} &= \nu \quad \forall i < N \\ -\frac{\partial v_N}{\partial m_N} &= \nu\end{aligned}$$

where λ_i is the multiplier for $v_i(p, m_i) \leq \overline{v}_i$ and ν is the multiplier for the homogeneity-zero constraint $\sum_{i\in\mathcal{N}} m_i = \sum_{k\in\mathcal{K}} p_k$. (Gradients are replaced with Clarke subdifferentials if an indirect utility is not differentiable.) Combined the first-order conditions imply

$$W + \sum_{i\in\mathcal{N}} \left(\frac{\partial v_i}{\partial m_i}\right)^{-1} \nabla_p v_i = 0$$

Now $v_i(p, m_i) = u_i(x_i(p, m_i))$ where consumer i's optimal demand $x_i(p, m_i)$ satisfies Roy's identity: $x_i(p, m_i) = -\nabla_p v_i / \partial_{m_i} v_i$. The first-order conditions thus imply

$$\sum_{i\in\mathcal{N}} x_i(p, m_i) = W$$

In other words, if $v = (v_1(p, m_1), \ldots, v_N(p, m_N))$ is a minimal element of the VPS then $v = (u_1(x_1(p, m_1)), \ldots, u_N(x_N(p, m_N)))$ where the optimal demands $x_i(p, m_i)$ for $i \in \mathcal{N}$ are feasible. Hence, v belongs to the UPS.

To show that any maximal element $u = (u_1(x_1), \ldots, u_N(x_N))$ of the UPS belong to the VPS recall that, for $x = (x_1, \ldots, x_N) \in F(W)$,

$$u_i(x_i) = \min_{\langle p_i | x_i \rangle = m_i} v_i(p_i, m_i) \tag{A.1}$$

If u is maximal then $\sum_{i\in\mathcal{N}} x_i = W$ and $\sum_{i\in\mathcal{N}} \langle p | x_i \rangle = \sum_{i\in\mathcal{N}} m_i = 1$. What we need to show is that if u is maximal then the price p_i that minimizes consumer i's indirect utility is the same for all $i \in \mathcal{N}$.

A standard envelope argument[14] shows that $\partial u_i / \partial x_{ik} = (\partial v_i / \partial m_i) p_{ik}$ if $x_{ik} > 0$

[14] From $v_i(p, m_i) = \max_{\langle p|x_i\rangle = m_i} u_i(x_i)$ we have $\partial v_i / \partial m_i = \lambda$ where λ is the multiplier of the budget constraint. Moreover, from $u_i(x_i) = \min_{\langle p|x_i\rangle = m_i} v_i(p, m_i)$ we have $\partial u_i / \partial x_{ik} = \lambda p_k$ if $x_{ik} > 0$ and $\partial u_i / \partial x_{ik} \leq \lambda p_k$ if $x_{ik} = 0$.

and $\partial u_i/\partial x_{ik} \leq (\partial v_i/\partial m_i)p_{ik}$ if $x_{ik} = 0$. Let $\text{MRS}^i_{kl} = \partial_{x_{ik}} u_i/\partial_{x_{il}} u_i$ denote consumer i's marginal rate of substitution. There exists a common price vector, p, that minimizes every consumer's indirect utility if, for $i \in \mathcal{N}$ and $k,l \in \mathcal{K}$,

$$\begin{cases} \text{MRS}^i_{kl} = \frac{p_k}{p_l} & \text{if} \quad x_{ik} > 0, x_{il} > 0 \\ \text{MRS}^i_{kl} \leq \frac{p_k}{p_l} & \text{if} \quad x_{ik} = 0, x_{il} > 0 \\ \text{MRS}^i_{kl} \geq \frac{p_k}{p_l} & \text{if} \quad x_{ik} > 0, x_{il} = 0 \end{cases} \tag{A.2}$$

Consider the program to obtain maximal elements of the UPS:

$$\max_{\substack{x \in F(W) \\ u_i(x_i) \geq \overline{u}_i \,\forall\, i < N}} \quad u_N(x_N)$$

The first-order conditions are

$$\begin{aligned} \lambda_i \frac{\partial u_i}{\partial x_{ik}} &= \mu_k - \nu_{ik} \quad \forall\, i < N \\ \frac{\partial u_N}{\partial x_{Nk}} &= \mu_k - \nu_{Nk} \end{aligned}$$

where λ_i is the multiplier for $u_i(x_i) \geq \overline{u}_i$, μ_k is the multiplier for the feasibility constraint $\sum_{i \in \mathcal{N}} x_{ik} \leq W_k$, and ν_{ik} is the multiplier for the non-negativity constraint $x_{ik} \geq 0$. We next show the first-order conditions imply (A.2). If $x_{ik} > 0$ and $x_{il} > 0$ then $\nu_{ik} = \nu_{il} = 0$ and $\text{MRS}^i_{kl} = \mu_k/\mu_l$. If $x_{ik} = 0$ and $x_{il} > 0$ then $\nu_{ik} \geq 0$ and $\nu_{il} = 0$ so $\text{MRS}^i_{kl} \leq \mu_k/\mu_l$. Finally, if $x_{ik} > 0$ and $x_{il} = 0$ then $\nu_{ik} = 0$ and $\nu_{il} \geq 0$ so $\text{MRS}^i_{kl} \geq \mu_k/\mu_l$. To summarize, the conditions in (A.2) are satisfied for $p = \mu$. ■

To prove the "Moreover" part, i.e. that the solution to (3) is unique for a given $m \in \Sigma_N$, suppose, in contradiction, that (p,m) and (p',m) are two solutions to (3). This means that both are minimal elements of the VPS. Strict quasiconcavity of the indirect utilities implies that, for $i \in \mathcal{N}$, $v_i(\frac{1}{2}(p+p'),m) < v_i(p,m)$ or $v_i(\frac{1}{2}(p+p'),m) < v_i(p',m)$, contradicting minimality of either (p,m) or (p',m). ■

Proof of Remark 1. We will establish existence of a maximal element of the UPS (the argument for existence of a minimal element of the VPS is similar). For $i \in \mathcal{N}$ and $k \in \mathcal{K}$, let $\omega_{ik} > 0$ and define $S_\omega = \{u \in \mathbb{R}^N | u_i \geq u_i(\omega_i) \,\forall\, i \in \mathcal{N}\}$. Feasibility and continuity of the utility functions ensure that $S_\omega \cap \text{UPS}$ is a compact set, called a *compact section* of the UPS. By Theorem 6.3.c in (Jahn, 2011), this compact section has a maximal element, which is also a maximal element of the UPS. ■

Proof of Theorem 2. For allocations that satisfy consumers' budget constraints, $u_i(x_i) \leq v_i(p, \langle p|\omega_i\rangle) = \max_{\langle p|x'_i\rangle = \langle p|\omega_i\rangle} u_i(x'_i)$ for $i \in \mathcal{N}$, so the program's value is non-positive. If (x,p) is a Walrasian equilibrium then allocations are optimal at price p, i.e. $u_i(x_i) = \max_{\langle p|x'_i\rangle = \langle p|\omega_i\rangle} u_i(x'_i)$, so (x,p) is a root, whence maximizer. Conversely, if (x,p) is a root of the objective then each term in the objective's sum is zero (as they are all non-positive and weights are positive). But $u_i(x_i) = v_i(p, \langle p|\omega_i\rangle)$ for $i \in \mathcal{N}$ means everyone is maximizing at price p and $x \in F(W)$ means the optimal demands are feasible. Hence, (x,p) is a Walrasian equilibrium. ■