Minimal Intelligence in the Double Auction: Logit-Choice and Reservation Utility

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Abstract

I propose a new boundedly-rational trader-behavior model for the continuous double auction, fit in the general equilibrium framework. Traders' order placement and acceptance strategies are driven by GE-adaptions of two classic mechanisms: within-period reservation prices (here utilities) (Friedman (1991)), and price-acceptability beliefs (Gjerstad and Dickhaut (1998)). Reservation utilities are considered from both a buying and selling standpoint. Reservation utility associated with the side of entry is used to gauge immediate acceptability of contra-side orders while contra-side reservation utility informs the trader of admissible orders for placement. Traders place logit choice probabilities over their set of admissible orders, based on beliefs formed about the acceptability of each order. The logit choice parameter allows the precision of trader order placement to range from uniformly random to perfectly maximizing given beliefs (mapping the range of trader behavior from Gode and Sunder (1993) to Gjerstad and Dickhaut (1998)). Simulations report market performance for this model, finding promising measures of efficiency and evidence of convergence to nearly Pareto optimal allocations.

Keywords: Continuous Double Auction, General Equilibrium, Exchange Economy, Logit Choice, Reservation Utility
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1 Introduction

Markets imbue agents with a centralized setting to reallocate commodities amongst themselves in an effort to arrive at a utility-improving state. Generally, the thought is that through markets this reallocation process is abundantly efficient. How agents in these markets traverse the landscape of even simple general equilibrium spaces such as an Edgeworth box is still to be determined to some extent. Many questions have been, are currently being and will continue to be formed about this process. This paper presents and investigates the following. Must every step on a markets path to equilibrium be utility-improving? Likely not, but what motivation guides traders to deviate and how can they do so in a productive manner?

Classical agent-based models have provided several mechanisms for how traders achieve equilibrating play and market convergence, though with nearly all assuming some form of utility-improvement restriction. Traders are presumed to perfectly move along the utility-improving side of their indifference curves (or, analogously, remain on the surplus improving side of their cost or redemption value schedules). Laboratory experiments, however, have repeatedly shown evidence, especially in more complex settings, that traders in a continuous double auction (CDA) routinely break this assumption. There are a few potential reasons for this. The mechanism privy to this paper's model is that traders occasionally intentionally take utility losing trades in order to set themselves up for future trades on both sides of the market.

Such a mechanism places the model in the 'wilderness of bounded rationality.' A vast expanse of deviations from perfect rationality have been explored, with many more yet to be charted. Two gates to this wilderness are typically recognized: that which assumes perfect rationality and that which imparts no (or very minimal) intelligence upon the economic agents. Models and implications at both entrances are numerous, though admittedly with a larger mass at the rational end. A mapping between gates, however, is less often attempted.

The main goal of this paper is to present a model that provides one such mapping. I propose a new agent-based model of CDA trader behavior in an Edgeworth box economy. The flavor of two influential assumptions on trader activity are incorporated: (1) beliefs on the acceptability of prices (Gjerstad and Dickhaut, 1998), and (2) reservation prices which adjust within-period (Friedman, 1991). Traders place orders by applying logit choice probabilities to each admissible[1] order; the logit choice parameter allows a trader's choice to capture precision anywhere between the random choice of zero-intelligence traders (Gode and Sunder (1993); Williams (2021b)) and the surplus-maximizing traders of Gjerstad and Dickhaut.

Section 2 presents a summary of past CDA trader behavior models. A wave of influential models at the end of the 20th century provide the design motivation for much of this paper's model (as well as a vast experimental and applied literature over the last twenty years). Advancements in behavioral learning models and a second wave of agent-based models of the double auction are also recounted. Section 3 proposes the setting of this paper. The double auction institution and general equilibrium setting are invoked to provide a test bed for the theorized agents.

A tractable model of trader behavior is presented in Section 4. The set of agents follow an order decision process and entry/re-entry rule that (1) loosen traditional restrictions on the placement of strictly-improving orders, (2) impart a more holistic view on market participation, and (3) incorporate the idea of adjusting an allocation for the sake of maneuverability (i.e. moving to a bundle with a more accommodating marginal rate of substitution for both directions of trade). Agents make use of time-dependent reservation utilities and logit choice to select and/or accept orders as two-way traders conscious of their positioning for both sides of the market. Section 5 proposes a simple design for a test of the model via simulations. Measures of efficiency are impressively high, which, when paired with convergence in allocations and prices, hints at relatively equitable reallocations near a point on the

[1] Nomenclature taken from Friedman (1991). Here admissible will mean satisfying a utility analog of reservation price.

contract curve. Three new or updated measures of efficiency are presented, as well as two penalized variants. The combination affords the ability to capture performance in utilities, prices and allocations and thus a fuller view on convergence. Section 6 concludes the paper.

2 Prior Theory

A few literatures are relevant for providing a background for this project, as well as framing its contribution overall. Theoretical work surrounding the continuous double auction price dynamics has arrived in two distinct waves over the last fifty years. First, a batch of partial equilibrium (PE) models were proposed in the late 80's and throughout the 90's. Then, a newer wave of more generally applicable models were given in the '00's and '10's. Here I'll present a synopsis of both waves, as well as a selection of related research in the intermediate period.

A trio of models set the scene for the first wave. Wilson (1987) proposed a game theoretic model, positing a strategic multilateral setting where each trader's actions directly impact the pricing strategies of the other traders. Easley and Ledyard (1993) took a less complex route to defining double auction play, entirely removing the strategic interaction. Traders participated in the market under the assumption their own decisions have no impact on the order placement/acceptance of others, all the while guiding their own orders via an across-period deterministic reservation price. Friedman (1991) similarly took on a game against nature stance, however with traders administering a more sophisticated within-period reservation price bidding/selling strategy. Two more influential models followed, providing bounded-rationality bookends for the wave. Gode and Sunder (1993) simplified trader behavior even further (hence its running name of "zero intelligence") by having order price be randomly chosen, supposedly leaving the only driving factor of price formation being the underlying rules of the double auction itself. Much closer to the perfectly rational gate, Gjerstad and Dickhaut (1998) models traders who develop beliefs on the acceptability of prices, and then select the price which yields the maximum expected surplus.

The literature from this point split over the last of couple decades. A batch of parsimonious, tractable, often heuristic-driven models entered the learning literature. Though not directly designed for markets, the following models can naturally be bent to account for the more complex setting. Roth and Erev (1995) provide a reinforcement learning model designed for dynamic games, with an emphasis on testing performance and convergence in the intermediate term. Agents develop choice propensities for each strategy, with successful outcomes increasing a strategy's propensity to be chosen in future decisions.[2] Fudenberg and Levine (1995) postulate a theory of 'cautious fictitious play', which places beliefs over the probability of opponent's playing given strategies. Agents use these beliefs to make their own strategy, each strategy being chosen with some logit choice probability.[3] Camerer and Ho (1999) house reinforcement-based learning and belief-based learning as special cases of a more complex experience-weighted attraction learning model; some flexible convex combination of the two is shown to generally be a better fit to game data than either of the two as stand-alone models.

A strain of models imposing higher levels of complexity in behavior or more complex market settings, or both, have also been proposed recently. One such model that is highly malleable in terms of its application and setting is the individual evolutionary learning model (IEL) of Arifovic and Ledyard (2011). Economic agents maintain an evolving pool of potential choices which they draw from subject to a probability distribution that is constantly updating via experimentation and replication stages. A few years later, Anufriev et al. (2013) applied IEL to the continuous double auction setting, in a partial equilibrium environment.[4] General equilibrium adaptions of the ZI model were promoted by Gode et al. (2004) and Crockett et al. (2008) a decade or so after the original model was published. The former features a price-angle order choice process, while the latter proposes a learning process by

[2] The model was tested across three game types, with one being a simple market.

[3] Feltovich (2000) tested these two models in the laboratory where subjects played a two-stage game with asymmetric information. The reinforcement model better predicted choice probability of a subject's next action, while the belief-based model proved better more often for aggregate trends in play.

[4] The timing in the paper lends itself to both an analysis of multiple iterations of a call market as well as a double auction in near continuous time. van de Leur and Anufriev (2018) extends the model via a more complex timing problem.

which the allowable subset of the contract curve is restricted round after round. Williams (2021a,b) bring two models from the first wave to a general equilibrium setting. An alternative model to that of Gode, Spear and Sunder (2004), Williams (2021b) postulates a GE-based zero intelligence (from here, ZI-G) model with a lower sense of 'zero' intelligence. Williams (2021a) (henceforth 'GD-G') brings the belief-based process of Gjerstad and Dickhaut (1998) to general equilibrium to better understand impacts price information may have on market convergence.

3 Environment

Consistent with Williams (2021a,b), I consider a market over two goods, X and Y. Traders in this market begin each trading day, or period, with some endowment of each of these goods, with the market totals of these endowments helping to define the Edgeworth box the traders are placed in. One of these goods is considered a standard commodity (let this be X), and the other considered as a numeraire (Y), as is the case with most if not all such settings since Shapley and Shubik (1967). Such an assumption allows the traders to trade quantities of X at prices represented by units of Y per unit of X in traditional auction settings.

3.1 Message Space

Here, I lay out the space encompassing all exchange related information the traders are given: the message space. This space is a crossing of several one dimensional sets (yielding information in the form of order n-tuples) to be described below.

Prices: $P \subset \mathbb{R}_+$ s.t. elements $p \in P$ denote per unit prices

Quantities: $Q \subset \mathbb{R}_{\setminus\{0\}}$ s.t. elements $q \in Q$ denote desired unit adjustment

Time: $\mathcal{T} \equiv [0, T] \subset \mathbb{R}_+$ s.t. elements $t \in \mathcal{T}$ denote time in a market period

The above sets create a 3-dimensional space which defines the standard elements of orders: the per-unit price, the desired adjustment in units of the commodity, and the time in the markets life at which the order was placed. In this paper, I assume orders are infinitely-lived, only expiring if the market's duration ends or if a trader replaces their order with a new one.

This typical 3-tuple order will be augmented to store information about the trader's involved. The set of traders, $\mathcal{N}$, is split into two subsets: natural buyers, $\mathcal{B} = \{1, ..., N_B\}$, and natural sellers, $\mathcal{S} = \{N_B+1, ..., N_B+N_S\}$. The set $\mathcal{N}$ satisfies $\mathcal{N} = \mathcal{B} \cup \mathcal{S}$ and $\mathcal{B} \cap \mathcal{S} = \emptyset$; $\|\mathcal{N}\| = \|\mathcal{B}\| + \|\mathcal{S}\| = N_B + N_S = N$. Natural buyers are traders whose marginal rate of substitution (MRS) at the inception of a market is greater than the competitive equilibrium (CE) price for the initial allocation of goods. Natural sellers are defined similarly, with their MRS residing below the initial CE price. Note that MRS changes throughout the life of the market, potentially enough to transition a trader from one side of the competitive equilibrium price to the other; accordingly, these names are just used as an indexing convention.

With trader identities defined, we can define the standard notation for orders in this framework:

$$o_{\Delta,a} = (p_a, q_a, t_a^{enter}, t_a^{exit}, b_a, s_a)$$

where a denotes the action number (the a^{th} action taken in this market), with $a \in \mathcal{A} \subset \mathbb{N}$; t_a^{enter} and t_a^{exit} are the times at which order $o_{\Delta,a}$ enters and exits the exchange; b_a and s_a denote the buyer and seller associated with the order. Actions are defined as an order placement or order acceptance. Also, Δ denotes the side of the market the order is being placed, with

$$\Delta = \begin{cases} b, & q > 0 \\ s, & q < 0 \end{cases}$$

For any ordering being placed in the exchange, b_a or s_a must be 0, as only one trader has engaged with the order at this point. Additionally, for book-keeping purposes, new orders sent to the exchange have $t_a^{exit} = 0$.

The sets of potential bids and asks can thus be defined as follows:

$$Buys: \quad \Omega_B := \mathcal{P} \times \mathcal{Q} \times \mathcal{T} \times \mathcal{T} \times \mathcal{N} \times \{0\} \equiv \{o_{b,a} : p_a \in \mathcal{P}; q_a \in \mathcal{Q}; t_a^{enter}, t_a^{exit} \in \mathcal{T}; b_a \in \mathcal{N}; s_a \in \{0\}\}$$

$$Sells: \quad \Omega_S := \mathcal{P} \times \mathcal{Q} \times \mathcal{T} \times \mathcal{T} \times \{0\} \times \mathcal{N} \equiv \{o_{s,a} : p_a \in \mathcal{P}; q_a \in \mathcal{Q}; t_a^{enter}, t_a^{exit} \in \mathcal{T}; b_a \in \{0\}; s_a \in \mathcal{N}\}$$

Note that b_a and s_a can lie in all of $\mathcal{N}$, as opposed to $\mathcal{B}$ or $\mathcal{S}$, as all traders have the capacity to place orders and trade on either side of the market (i.e. two-way traders). The set of orders is defined as $\Omega \equiv \Omega_B \cup \Omega_S$.

3.2 Exchange Definitions

Every message which enters the market, or exchange, lives in the message space laid out above. Below, I define the characterizing elements/processes that make up the exchange.

Orders: An order is a single message sent to the exchange by a trader, of the form $o_{\Delta,a}$. Orders may be submitted at any time $t \in \mathcal{T}$.

Asks/Bids: Asks satisfy $q_a < 0$, $b_a = 0$, $s_a \neq 0$ s.t. $o_{s,a} \in \Omega_S$. Bids similarly satisfy $q_a > 0$, $b_a \neq 0$, $s_a = 0$ s.t. $o_{b,a} \in \Omega_B$.

Orderbook (Ω_O): The set of orders which have been sent to the exchange and actively satisfy $b_a = 0$ or $s_a = 0$, and $t_a^{exit} = 0$.

To accommodate the idea of trading units or filling orders, an amendment to the notation of orders is needed, as well as a definition of how orders are filled. An order, $o_{\Delta,a}^{\kappa}$, has $\kappa = 0$ if the order is newly posted to the orderbook, $\kappa \in (0, 1]$ if the order was (partially) filled. The value of κ in the latter case is equal to the proportion of units that were filled out of q_a. If $\kappa = 1$, order $o^{\kappa}_{\Delta,a}$ has been fully filled, and thus is removed from the orderbook.

Cross/Accept: An order $o_{\Delta,a}$ crosses order $o_{\Delta',a'}$ (for $a' < a$) if:

1. $\Delta \neq \Delta'$ and
2. $p_a \leq p_{a'}$ if $\Delta = s$ or $p_a \geq p_{a'}$ if $\Delta = b$

Also, define $a_T : \mathbb{N} \to \mathcal{A}$ as the mapping from the ordering of crossings/traders to the action at which the crossing/trade occurred.

3.3 Histories

While the orderbook provides a snapshot of the present state of the exchange, a system for (1) referencing older orders no longer in the book, and (2) providing context for the expanse of trader's memories within the market must be defined to track the adjustment of the market.

History: The set of all orders (past and present) and trades in the lifetime of the market. The full history can be split into three types of orders:

- Trades (Ω_τ) $\longrightarrow$ The set of orders $o^{\kappa}_{\Delta,a}$ which satisfy $b_a, s_a \in \mathcal{N}$ and $\kappa = 1$.
- Cancelled Orders (Ω_C) $\longrightarrow$ The set of orders $o^{\kappa}_{\Delta,a}$ which satisfy $b_a = 0$ or $s_a = 0$, and $\kappa = 1$
- Orderbook (Ω_O; as defined above)

A useful union of these subsets for the purpose of understanding trader behavior is $\Omega_T \cup \Omega_C$, which houses all past, or closed, orders in the history of the market. Also, note that this union is the definition of *history* (Ω_H) in Gjerstad and Dickhaut (1998).

The history, as just defined, shows the entirety of the exchange's life, from the perspective of the exchange (in the sense that the history is full/complete). Each trader, however, may or may not have the capacity (or desire) to maintain a complete record of the history. In the vein of Gjerstad and Dickhaut (1998), traders can remember all past orders within the last L successful trades:

Memory: The set of traded orders and cancelled orders that have occurred in the last L orders (where the most recent trade was the j^{th} trade in the market) , $\Omega_{M(L)} \equiv \{o^{\kappa}_{\Delta,a} \in \Omega_H : a > a_T(j-L)\}$.

3.4 Trader Preferences

Much like the ZI-G and GD-G general equilibrium models, traders are motivated via utility functions. This is opposed to the cost and redemption-value schedules driving traders in more classical partial equilibrium settings. Generally, the standard assumptions on the utility function of trader i, u_i, are assumed: u_i is twice differentiable and quasi-concave. For the remainder of this paper, I'll focus on the constant elasticity of substitution functional form:

$$u_i(x,y) = c_i((a_i x)^r + (b_i y)^r)^{\frac{1}{r}} \tag{1}$$

For simplicity, I normalize relative preference parameters a and b such that they sum to one and are both non-negative. The curvature parameter r is also assumed to lie in $(-\infty, 1]$ to satisfy the quasi-concavity requirement.

I add the flavor of reservation prices to trader's preferences, though through an avenue more appropriate for general equilibrium. Traders maintain reservations around the utility gained at each price (or quantity change in x). As agents are two-way traders, they develop these reservations as both buyers and sellers. For example, an agent who enters as a buyer may set a reservation utility by considering an altered version of her utility function, one which now requires more X to satisfy a utility-improving order. Similarly an agent electing

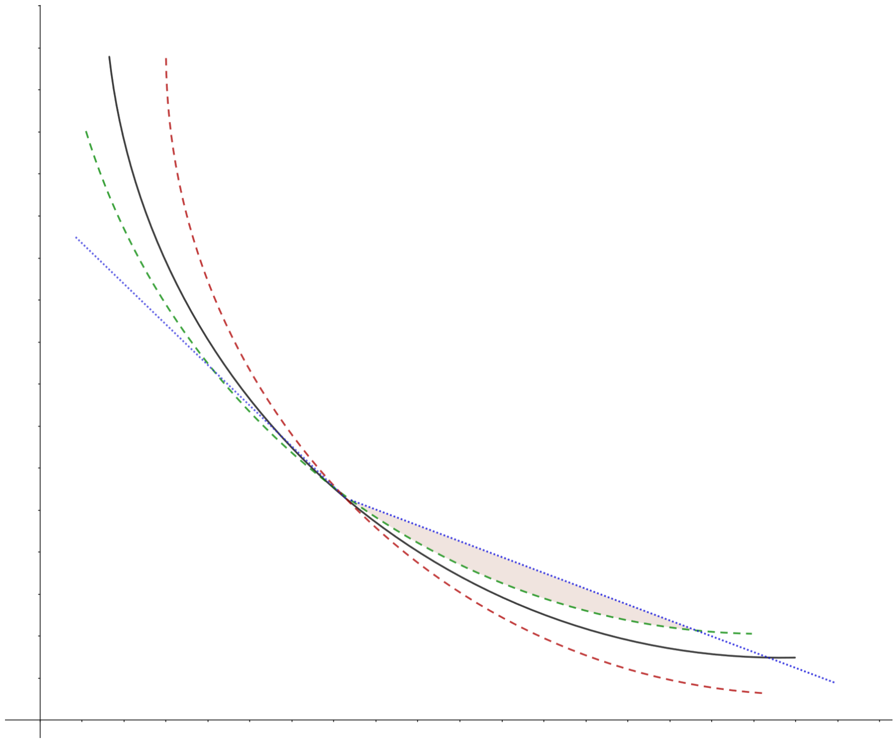

Figure 1: Reservation Utility. The black curve shows an indifference curve (IC) of u_i. The green and red dotted curves show IC's for $u_{i,b}$ and $u_{i,s}$, respectively. The two blue dotted line segments show best ask (right) and bid (left) prices in the market. The shaded region shows the space in which this trader would automatically accept a posted ask, were he to enter as a buyer.

to sell may consider an adjusted utility function which readjusts her relative preferences to favor Y less than her true preferences. For CES preferences, this would mean adjusting the relative sizes of a_i and/or b_i.

One natural consideration would be to have a piece-wise, kinked indifference curve satisfying the above adjustments. Such a model would be an attempt at a GE-version of the reservation price model of Friedman (1991). While this is an interesting option, explored in Appendix B, I take the position that this may be a waste of the 'other halves' of these adjusted indifference curves. Agents, instead, use each of these preference sets for two different order choice rules.

Figure 1 provides an example of the adjustment in curvature, and thus relative preferences over X and Y. The two rules, to be discussed at length in Section 4, dictate accepting orders and placing orders. Depending on the side of entry, the reservation utility IC above the true IC is used for the acceptance rule and the reservation IC below the true IC (the portion thrown out usually) guides the order placement rule.

These reservation utilities are captured via parameter η, where $\eta(t)$ is a function of within-period time t, and enters into the trader's utility function as follows

$$u_{i,b}(x,y|\eta) = c_i((a_i - \eta)^r x^r + (b_i + \eta)^r y^r)^{\frac{1}{r}} \tag{2}$$

$$u_{i,s}(x,y|\eta) = c_i((a_i + \eta)^r x^r + (b_i - \eta)^r y^r)^{\frac{1}{r}} \tag{3}$$

Here $u_{i,b}$ is the buyer reservation utility for trader i, and $u_{i,s}$ is the seller reservation utility. A few desirable statics arise when determining an appropriate functional for for η. First, as gains from trade decline over the life of the market and a smaller range of prices become competitive or desirable, $d\eta/dt < 0$ should be satisfied. Second, the size of the adjustment should be related to the trader's relative preference between the two goods. Namely, a trader who strongly prefers one good to the other may be less inclined to consider large deviations in their reservations away from their true preferences. Third, a trader's reservation preferences shouldn't change her outlook on a product from a 'good' to a 'bad', i.e. $\eta \leq min\{a_i, b_i\}$. Thus, the form of $\eta(t)$ I consider in this paper is

$$\eta(t) = \left(\frac{T-t}{T}\right)\frac{min\{a,b\}}{max\{a,b\}}min\{a,b\} \tag{4}$$

A few special cases should be mentioned as well. First, with respect to functional form, perfect substitutes ($r = 1$), perfect complements ($r \to -\infty$) and Cobb-Douglas ($r \to 0_{-,+}$) are naturally folded into CES preferences. Both perfect substitutes and perfect complements provide interesting responses/interpretations when including η as in equations 2 and 3. The former, graphically, mimics reference prices from the partial equilibrium literature of the 90's, as the slope of the IC gives a natural reservation price. The latter is analogous to an adjustment in the desired complement ratio.

4 Agent-Based Model

This section lays out the details of the model, now that the environment has been established. Much like the GD-G model from Williams (2021a), four main processes determine the flow of the market and trader behavior in this model. These are *entry*, *belief updating*, *market interaction*, and *re-entry determination*.

Entry refers to the actions taken and snapshot of the market received by the trader who enters the market in time t. In all times aside from the inception of the market, entry is actually the second step of a two-part market entry/exit flow process along with the re-entry determination phase. The *belief-updating* phase takes the snapshot of the market in the entry phase and allows the entrant to readjust his interpretation of which prices my potentially be successful moving forward. *Market interaction* defines the order selection and submission process, as well as potential clearing. The *re-entry determination* phase sees all traders briefly evaluate their holdings, beliefs and the state of the market to evaluate their desire for re-entry. Below, each of these will be fleshed out in much greater detail.

4.1 Entry

Entry (and re-entry) into this environment's markets can take a couple of different forms depending on the age of the market and the potential entrant's previous participation in the market. The inception of the market (i.e. the first entry in the first iteration, or period, of the market) is unique in that no prior history exists. As such, this is the only instance in which entry is entirely random. Similarly, the first entrant of any period after the first is uniformly drawn.

The second (and far more common) entry situation is any entry after the first in any market period. To foreshadow the re-entry process discussed in section 4.4, the trader who wins the re-entry draw (with re-entry probabilities being dependent on average utility gain above a given trader's reservation utility) enters the market next. In this case, the trader drawn to (re)enter checks the market's best bid and ask against their own current reservation utilities and begins the belief updating process before making a decision on how they wish to use their entry.

4.2 Belief Updating

First, recall the belief formation and updating process of Gjerstad and Dickhaut (1998). Here, traders establish beliefs over the acceptability of certain prices on either side of the market. Traders recall a portion of the history, Ω_H, and tally the success and failure rate of each price, ρ, seen for each side of the market, $TA(\rho)$ for asks and $TB(\rho)$ for bids.

In Gjerstad and Dickhaut's original setting, these tallies were defined as counts with a count of 1 given to each order that satisfied the criteria (traded or cancelled) of interest. This was appropriate as each order in their partial equilibrium setting was required to be for a single indivisible unit. However, uniform counts are not attuned to settings with multiple and/or divisible units. Williams (2021a) provides a general-equilibrium-adjusted version of Gjerstad and Dickhaut's model, in which each order is given weight equal to the proportion of the original quantity successfully traded, $\sqrt{q_k}\frac{q_{k,traded}}{q_k}$. A similar weighted count is defined for the rejected (cancelled) portions of orders, $RA(\rho)$ and $RB(\rho)$.

Traders aggregate over the success of orders at less desirable (to the rest of the market) prices than one they may be considering. This is assessed relative to the success of these worse prices along with the failure of prices placed on the desired side at more desirable prices. In the notation of Gjerstad and Dickhaut (1998), for some bid ρ,

$$p_b(b) = \frac{\sum_{\rho \leq b} TB(b) + \sum_{\rho \leq b} TA(b)}{\sum_{\rho \leq b} TB(b) + \sum_{\rho \leq b} TA(b) + \sum_{\rho \geq b} RB(b)} \tag{5}$$

represents the probability of acceptance. The analogous definition for sell price acceptability is

$$p_s(s) = \frac{\sum_{\rho \geq s} TA(s) + \sum_{\rho \geq s} TB(s)}{\sum_{\rho \geq s} TA(s) + \sum_{\rho \geq s} TB(s) + \sum_{\rho \leq s} RA(s)} \tag{6}$$

Each trader holds such a belief for each price represented in Ω_H. Note that beliefs are over the domain $[0, M]$, with $p_b(0) = 0$ and $p_b(M) = 1$ for bids and the reverse for asks.

4.3 Market Interaction

Contrary to ZI-G and GD-G, this model considers the use of two types of order placement strategies, accepting orders directly and placing orders in the book. While both of the prior models can achieve both strategies via only the latter (as crossing orders essentially accept another order directly), a couple of distinctions should be made. First, in a setting where orders can have multiple and/or partial unit quantities, crossing orders won't always interact as cleanly in the orderbook as an accept. Second, it seems natural to consider the two actions as responding to separate lines of intent for the trader, with accepts being very short-term, heuristic driven choices and orderbook additions being more long-term plays. Establishing such distinctions between the two also provides a nice analog to the ideas of market orders and limit orders in the financial literature.

The market interaction, in concert with the above, is a two part process: checking for and interacting with orders that may be desirable immediately, and submitting an order to the exchange to add to the existing book. Note that the second step is only reached if the trader does not satisfy the "interacting" portion of the first step. Below are the processes of accepting and placing orders explained in detail.

4.3.1 Accepting Orders

Upon entry, even before the belief updating process has occurred, the trader has an idea of their reservation utility on their selected side of entry. Consistent with previous reservation price models, traders have an incentive and desire to accept with certainty an order on the contra-side of the market whose price is better than their reservation. This means the trader would be checking first if the current book leaves any room between the best order on the entered side, $BP_{-\Delta}$, and his current reservation utility, or:

$$|BP_{-\Delta} - MRS_{u_i}| - |MRS_{u_{i,\Delta}} - MRS_{u_i}| > 0 \tag{7}$$

A check with evidence of a contra-side order in this region induces the entrant to accept the order outright. If multiple orders exist in this region, the order with the highest resultant utility is chosen. A null result from the check leads the trader to stage two of their market interaction. Figure 1 displays such a check, where an entry on the buy side could yield an auto-accept as the best ask price vector lies above the stricter reservation utility. If the quantity associated with the ask yields a utility improvement, then the trader will accept the order.

4.3.2 Placing an Order

While on side Δ, the trader has three indifference curves to consider: the curve for $u_{i,\Delta}$, the curve for u_i and the curve for $u_{i,-\Delta}$. Functionally, only two of these will be considered. The more restrictive reservation utility, $u_{i,\Delta}$, has already been shown to be used as a bound for immediately-acceptable orders. The weaker reservation utility, $u_{i,-\Delta}$, provides a lower bound for the bundles necessary to be at least as happy in future entries (especially if entering on side $-\Delta$ in their next interaction). The curve associated with u_i is left to serve as a target for activity very late in the market's life, with $u_{i,\Delta}$ and $u_{i,-\Delta}$ providing "goal posts" moving over time as a reflection of a trader's continuation value.

Using this lower goal post as a criterion for utility-improving orders, the trader considers any bundle on the Δ side that is weakly better than their current reservation utility $u_{i,-\Delta}$. This set of orders lies in

$$\mathcal{P}_{u_{i,-\Delta}} \times \mathcal{Q}_{u_{i,-\Delta}} \equiv (min\{|MRS_{u_{i,-\Delta}}|, Boundary_\Delta\},\ max\{|MRS_{u_{i,-\Delta}}|, Boundary_\Delta\}] \times (0, \bar{x}(\Delta)] \tag{8}$$

$\mathcal{P}_{u_{i,-\Delta}}$ takes an open lower bound at the marginal rate of substitution at the trader's current endowment on the contra-side reservation utility $u_{i,-\Delta}$ and a closed upper bound at the boundary price on that side ($Boundary_\Delta$; 0 if $\Delta = b$ and M if $\Delta = s$). $\mathcal{Q}_{u_{i,-\Delta}}$ is more tedious to define, as the upper bound must take both current allocation and the non-zero[5] intersection point between the indifference curve and line associated with the best price on that side, BP_Δ, into account. The upper bound on $\mathcal{Q}_{u_{i,-\Delta}}$ is dependent on both the intersection between $u_{i,-\Delta}$ and the price vector extending from the trader's current allocation (call this $\hat{x}$) and the trader's current holdings of x. When $\Delta = b$, $\bar{x}$ is generally equal to $\hat{x}$; however, if $\hat{x}$ is non-existent or sufficiently large, then $\bar{x}$ is bounded above by the total x remaining in the market. For $\Delta = s$, $\bar{x}$ is the minimum of the total y remaining in the market divided by the price of the order and $\hat{x}$.

For each potential bundle, $o_z \in O_z := \mathcal{P}_{u_{i,-\Delta}} \times \mathcal{Q}_{u_{i,-\Delta}}$, the trader considers their belief on the acceptability of the given price. Each bundle thus has an expected level of utility improvement. The trader considers the possible bundles with logit choice probability:

$$Pr(o_z|x_k, y_k, \Omega_M) = \frac{exp[\lambda p_\Delta(o_z) u_{i,-\Delta}(o_z))]}{\sum_{o'_z \epsilon O_z} exp[\lambda p_\Delta(o'_z)(u_{i,-\Delta}(o'_z))]} \tag{9}$$

The parameter λ implies some preciseness over the trader's ability to choose the expected utility-gain maximizing order. Reservation adjustment aside, $\lambda = 0$ would yield a uniform distribution over the orders, much like ZI-G. Similarly, $\lambda \to \infty$ would imply perfect choice as in GD-G.

4.4 Re-entry Determination

Now that the current entrant has entered, updated and (attempted to) place their order, and the exchange has updated the book and/or processed a transaction, the rest of the market (and the entrant herself) can individually reflect and gather their potential gains on either side of the market. For side Δ, each trader considers $u_{i,\Delta}$ when determining the set of admissible orders (those which are immediately acceptable) $\mathcal{P}_{u_{i,\Delta}} \times \mathcal{Q}_{u_{i,\Delta}}$. Each

[5] The "zero" intersection here would be at the trader's current allocation.

admissible order is given an expected utility gain using the trader's developed beliefs for price acceptability. The trader averages over the expected gains of all admissible orders, giving them an idea of the expected gain for entering on that side. Each trader-side is treated as a separate draw for the next entry into the market, with each draw's probability being the draw's expected gain divided by the sum of all trader-side expected gains.

5 Simulations

5.1 Implementation

The performance of the model presented here is demonstrated via a set of simulated markets. A group of eight computerized traders are placed in a simulated CDA, playing in multiple periods of a single market. This multi-period-life market is simulated many times, completely refreshed at the inception of each simulation.

The main assumptions of the model, institution and equilibrium are applied to the traders; a series of 40 markets are simulated under these conditions (and with the parameters described below). Each market lives twelve periods of identical length. A market period is comprised of 200 market entries, with the entrant being determined via the draw described in Section 4.4.[6]

Each computerized trader has CES preferences over two goods, with parameter sets:

	c	a	b	r	(x_{Endow}, y_{Endow})
Buyers	0.113	0.825	0.175	0.5	(3,23)
Sellers	0.099	0.6875	0.3125	0.5	(11,3)

Table 1: Simulated Agent Parameters.

[6]As in the model, no market level spread reduction rule is enforced. However, traders have 'internal' spread rules, only replacing their own order if its better than one currently in the market. As this still allows for order placement at prices worse than the best bid and ask, I don't feel such a restriction is overly influential in market success. These internal rules are the only impediment on orders not being placed in the book. In the batch of simulations discussed here, 146 out of 200 orders were placed per period on average. See Appendix A.5 for simulations without such a reduction rule.

Given these parameters, the natural buyer's and seller's η adjustments begin at 0.035 and 0.142, decaying to 0 by the end of each period. The competitive equilibrium price associated with the trader's true preferences, u_i, at the endowment point is 2.44. If trader's trade solely based on their stricter preferences, $u_{i,\Delta}$, the CE price calculated at time 0 and for the starting endowments is 2.82.[7] Following only lower reservation utilities, $u_{i,-\Delta}$, the starting CE price would be 2.24.

All traders maintain a memory of $L = 5$, implying they can perfectly recall all transactions and order cancellations in the market within the last five transactions.[8] This memory may span across market periods within the same run, however may not carry over between runs.[9] Additionally, when conducting the logit choice procedure over the set of feasible orders, each trader will have a logit choice parameter, λ, of 5. This places traders' choice precision between uniformly random and perfect, leaning more on the side of random. See Appendices A.1 and A.3 for robustness runs testing history and λ choices.

5.2 Measuring Efficiency

Traditionally, the measure coined as 'allocative efficiency' has been used as the main efficiency measure in this type of simple market. The measure comes from the partial equilibrium literature, defined as the sum of profits made by each of the traders on each of their units over the expected gain across all units in equilibrium. More formally, this is written as

$$\frac{\sum_{b=1}^{B}\sum_{i=1}^{P_b}(p_{i,b} - v_{i,b}) + \sum_{s=1}^{S}\sum_{j=1}^{P_s}(c_{j,s} - p_{j,s})}{\sum_{b=1}^{B}\sum_{i=1}^{P_b}(p_{CE} - v_{i,b}) + \sum_{s=1}^{S}\sum_{j=1}^{P_s}(c_{j,s} - p_{CE})} \tag{10}$$

where B and S are the cardinalities of the buyer and seller sets, and P_b and P_s are the number of buyer and seller units at the inception of a market. The sets $\{v_i\}$ and $\{c_j\}$ are

[7] This falls to 2.59 at the halfway point of a period.

[8] This is the memory length used in Gjerstad and Dickhaut (1998). I test two other memory lengths, 0 and 10, book-ending this choice for robustness. See Appendix A.2.

[9] This assumption is tested in a batch of simulations with trader memories that refresh at the beginning of every period.

the buyers' and sellers' redemption values and unit costs, while $\{p_{i,b}\}$ and $\{p_{j,s}\}$ are the buy and sell trade prices and p_{CE} is the competitive equilibrium price.

One natural analogue to this measure in GE is to replace the main PE outcome, cash profit, with the main GE outcome, utility gain. As such, I term the sum of realized utility gains over expected utility gains, seen in equation (11), as allocative efficiency in a two-good Edgeworth box economy.

$$E^{Alloc} \equiv \frac{\sum_{i=1}^{N}(u_i(x_{i,T}, y_{i,T}) - u_i(x_{i,0}, y_{i,0}))}{\sum_{i=1}^{N}(u_i(x_i^{CE}, y_i^{CE}) - u_i(x_{i,0}, y_{i,0}))} \tag{11}$$

While this provides a nice outlook on the gains from trade reaped by the market in utility terms, the measure is undiscriminating in terms of relative gains across individual traders. For example, for two market realizations both not reaching competitive equilibrium, it is entirely possible the two vectors of utility gains are noticeably different. In fact, the two vectors need not even have equivalent lengths. I provide a penalized version of allocative efficiency as well, where the value in (11) is multiplied by a scaling penalty defined in (12). For interpretation of the following notation, $\vec{v}_{\omega\to T}$ is the vector from the tuple of utilities at a period's starting endowment to the tuple of final utilities at time T in the period and $\vec{v}_{\omega\to CE}$ is the vector from the tuple of utilities at a period's starting endowment to the tuple of expected utilities in competitive equilibrium.

$$\frac{\|proj_{\vec{v}_{\omega\to CE}}\vec{v}_{\omega\to T}\|}{\|\vec{v}_{\omega\to T}\|} \quad \text{where} \quad proj_{\vec{v}_{\omega\to CE}}\vec{v}_{\omega\to T} \equiv \frac{\vec{v}_{\omega\to T}\cdot\vec{v}_{\omega\to CE}}{\|\vec{v}_{\omega\to CE}\|^2}\vec{v}_{\omega\to CE} \tag{12}$$

Thus, if the tuple of final utilities deviates greatly from the path determined by $\vec{v}_{\omega\to CE}$, the estimate is scaled down proportional to its deviation.

I also provide a second interpretation of partial equilibrium's allocative efficiency in general equilibrium, named 'profit efficiency'. For each trade made by a pair of traders, each trader i has a price, call it $p^{Ind}(i)$, at which she could have moved the same number of units of X on the side she traded while remaining on the same indifference curve. The difference in this price and the actual trade price, multiplied by the traded quantity,[10] is the GE analog of a cash gain, i.e. the gain (or loss) in the numeraire. Such a gain summed across all trades across all traders divided by the gain when replacing each trade price with the CE price defines this profit efficiency, as in (13). For notation, let A_τ be the set of actions a resulting in a trade, i.e. the a represented by the set of orders o comprising Ω_τ.

$$E^{Profit} \equiv \frac{\sum_{a\in A_\tau}[\kappa_a q_a(p^{Ind}(b_a) - p_a) + \kappa_a q_a(p_a - p^{Ind}(s_a))]}{\sum_{a\in A_\tau}[\kappa_a q_a(p^{Ind}(b_a) - p_{CE}) + \kappa_a q_a(p_{CE} - p^{Ind}(s_a))]} \tag{13}$$

Though these two efficiency measures provide a meaningful account of the market's ability to capture gains from trade in utility and price terms, neither seems to capture the market's path and proximity to the equilibrium allocation. The third measure of efficiency I examine, 'distance efficiency', aims to capture the market's performance in allocations. Equation (14) defines the statistic as a deviation from one hundred percent. The deviation is measured as the distance to the equilibrium allocation bundle,$\{x^{CE}, y^{CE}\}$, at the end of a trading period relative the total distance traveled in equilibrium.

$$E^{Dist} \equiv 1 - \frac{d_{CE}(\{x^T, y^T\})}{d_{CE}(\{x^\omega, y^\omega\})} \tag{14}$$

with distance measure

$$d_{CE}(\{x, y\}) \equiv \frac{1}{N}\sum_{i=1}^{N}\sqrt{(x_i - x_{CE})^2 + (\frac{1}{p_{CE}}(y_i - y_{CE}))^2} \tag{15}$$

The endowment bundle and final bundle are denoted as $\{x^\omega, y^\omega\}$ and $\{x^T, y^T\}$.

Much like with the allocative efficiency measure, this measure may not fully account for large variation in allocative gains across traders. Thus, I provide a penalized version of distance efficiency as well. In line with the notation of the penalized allocative efficiency, I denote the vector from the endowment bundle to the equilibrium bundle as $\vec{\alpha}_{\omega\to CE}$ and the

[10]For a trade occurring in action a, this is $\kappa_a q_a$, or the proportion of the order filled multiplied by the quantity desired.

vector from the endowment to the final allocation as $\vec{\alpha}_{\omega\to T}$. As the set of bundles that satisfy the same level of distance efficiency all lie on the surface of the same ball around $\{x^{CE}, y^{CE}\}$, the penalty should increase in intensity as the final bundle deviates from $\vec{\alpha}_{\omega\to CE}$ and as the market begins over-trading. The penalty is thus defined as in equation (16), and is applied as a scaling term multiplying the distance efficiency measure in practice.[11]

$$\frac{\|\vec{\alpha}_{\omega\to CE}\| - \|\vec{\alpha}_{T\to CE}\|}{\|proj_{\vec{\alpha}_{\omega\to CE}}\vec{\alpha}_{\omega\to T}\|} \tag{16}$$

Notice the numerator refers to the length of the vector from the endowment bundle to the point lying on $\vec{\alpha}_{\omega\to CE}$ which is equidistant from $\{x^{CE}, y^{CE}\}$ as $\{x^T, y^T\}$.

5.3 Performance

Table 2 records the main performance measures for the simulated markets. A quick glance shows evidence of surprisingly successful markets. Estimates show promising levels of convergence in both allocation and price space, with markets tracking remarkably well around the equilibrium path.

All estimates are means of round-average (in the case of all price measures) or round-end (in the case of allocation and efficiency measures) level observations. Average price lies just 0.16 above the CE prediction from market inception, which, when accompanied with a relatively low average deviation, implies transaction prices lie in a tight band around the CE price. Figure 2 confirms not only the round-averages, but the individual transaction prices across the markets are closely bound. The per-unit average price (total units of y traded divided by total units of x traded across the period) is less sensitive to high outlier prices as they are accompanied with small trade quantities; de-weighting as such yields an estimate just 0.01 unit away from CE. Convergence within period, however, requires tighter bounds on the time in focus. The final batch of transactions in a period provide an idea of the traders' desire to trade and urgency to reap more gains from trade. I find an estimate

[11] $\|\cdot\|$ here is the standard Euclidean norm.

	Mean	St. Dev.	Range
I. Prices			
Price	2.60	0.22	(2.07, 3.51)
Per-Unit Avg.	2.43	0.26	(1.83, 3.55)
$\|Price - CE\|$	0.55	0.17	(0.24, 1.34)
RMSE	0.95	0.34	(0.31, 2.10)
Final 5 Prices	2.47	0.14	(2.13, 3.30)
II. Allocations			
Final Distance	0.86	0.58	(0.03, 3.79)
Seller MRS	2.41	0.17	(1.56, 2.79)
BuyerMRS	2.54	0.20	(2.18, 3.80)
III. Efficiencies			
Allocative	0.99	0.01	(0.91, 1.00)
- Penalized	0.77	0.10	(0.43, 0.97)
Distance	0.82	0.05	(0.65, 0.95)
- Penalized	0.68	0.08	(0.41, 0.88)
Profit	0.91	0.08	(0.67, 1.13)
Observations	480	480	480

Table 2: Simulation Outcomes. Observations at the round-average level. Panel I shows price related estimates. RMSE is the root-mean-squared error. Panel II reports outcomes in allocation space. MRS here is the marginal rate of substitution at the final allocation of aggregated representative agents. Panel III lists estimates for three measures of efficiency.

even tighter to the CE prediction, suggesting prices not only lie close to the equilibrium, but tighten and converge in some smaller bound as the period ends.

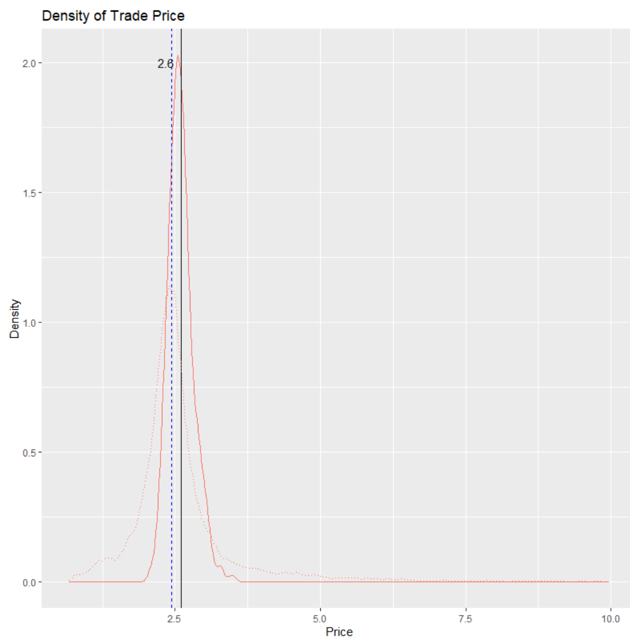

Figure 2: Kernel density for prices. Red line shows round-averages, while red-dotted shows individual transaction prices. The black vertical line is the mean of the round-averages, and the blue dotted line is the CE price of 2.44.

Even so, prices can only reveal a portion of the full success of the market. Panel II gives two distinct pictures of how these simulated traders reallocate the two goods among themselves. The first is how far away the market is as a whole from the equilibrium set of allocations. To examine this, I collapse[12] the two types of traders into representative agents. These agents can aptly be represented in the Edgeworth box. On average, the final distance[13] the pair lies away from equilibrium allocation bundle pair is within a unit radius of the final. Allocations approach the contract curve, on average lying in nearly Pareto optimal final resting places. As displayed in Figure 3, the geometric mean of the final allocations is

[12]For example, the four natural buyers can be aggregated into a single agent by averaging over each transaction made by one (or two) of the traders. If a buyer transacts with a natural seller, the adjustment in the representative buyer's allocation will be a quarter of that realized by the individual trader. If two natural buyers transact, the representative sees no adjustment in his allocation.

[13]I.e. the Euclidian distance that the representative buyer (and equivalently, seller) is away from the equilibrium allocation in the Edgeworth box. The y contribution to the distance is de-weighted by the equilibrium price. The distance function is thus $dist(\cdot) = \sqrt{(x_i - x_{CE})^2 + (\frac{1}{p_{CE}}(y_i - y_{CE}))^2}$.

very close to the equilibrium bundle, and even closer to the contract curve in general. In fact, the vast majority of the final allocations lie on the contract curve or quite close.

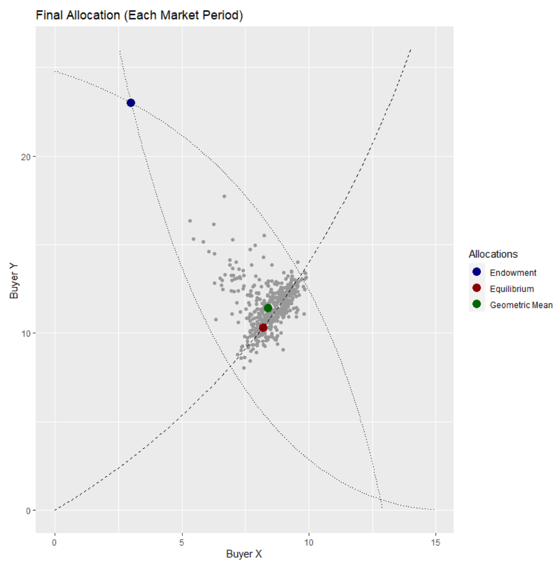

Figure 3: Final Allocations. Each grey dot represents the final allocation of the representative agents in the Edgeworth box. The red dot shows the equilibrium bundles, while the green dot represents the geometric mean of the scattered grey dots. The CE-price de-weighted distance between the red and green dots is 0.478 units. The dotted lines show the indifference curves of the representative agents evaluated at the endowment allocation. The dashed line shows the set of Pareto optimal allocations.

The marginal rate of substitution of market participants gives a proxy for convergence in allocative efficiency, as a trader's MRS should equal the CE price in equilibrium. Natural buyers are characterized by their initial MRS being above the equilibrium prediction; natural sellers lie on the other side of the price. As such, the traders, and their representative agents, should reallocate resources throughout the market period to collapse their MRS to the CE-price. The average final allocations of the representatives approach encouragingly close to 2.44, with sellers 0.03 below and buyers 0.08 above. Despite large ranges of round-end estimates for this spread, tight standard deviations suggest poorer MRS spreads are rather uncommon. Figure 4 reinforces such a claim, as around 90% of the periods yield an MRS spread within 0.5 units.

Three measures of efficiency are estimated: allocative, distance and profit. Each is meant to capture the performance of the market in a different convergence indicator: utility,

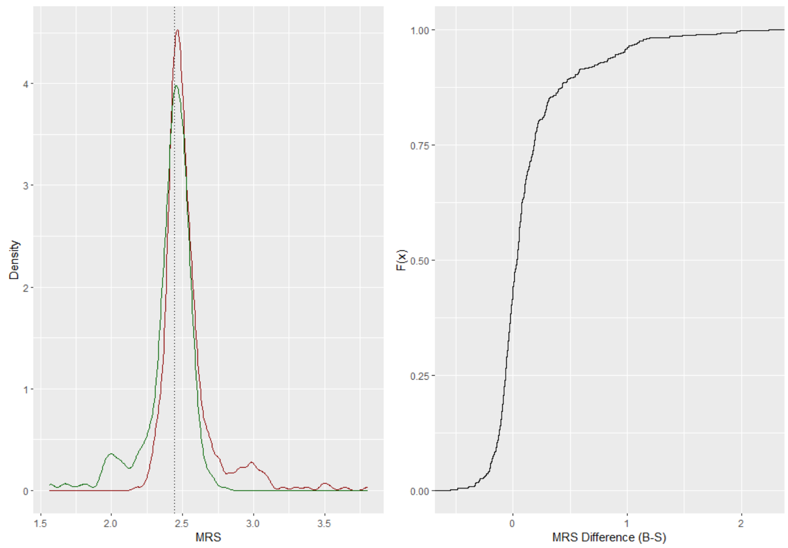

Figure 4: The left figure shows the densities for round-end representative agent Buyer (red) and Seller (green) MRS. The black line represents the CE price. The right figure shows the CDF for the round-end gap between the Buyer and Seller MRS.

allocations, and prices. The simulations report promising results in each.

First, the base measure of allocative efficiency reports an average of 0.99 with a minimum estimate of 0.91. The market clearly is capturing essentially all of the gains from trade achievable in utility-terms. However, is this being driven by one or two traders dominating the market or are gains seen by all traders? The penalized average of 0.77 reflects some mild deviation from the path given by $\vec{v}_{\omega \to CE}$, with the minimum estimate drops from 0.91 to 0.43. A Spearman rank correlation of 0.35 between the two measures suggests high efficiency periods are not entirely depend on markets with over equitable or inequitable utility gain distributions.

While gains from trade in utility terms are mostly being realized, is the market arriving at the correct final allocation? Figure 3 provides a hint in two dimensions via the representative agents. The base measure for distance efficiency corroborates these findings in space represented all traders individually, achieving an average value of 0.82. While an 18% average deviation may seem large, the localization along the contract curve in Figure 3 suggests these deviations are likely lateral. The penalized estimate of 0.66 supports this, as the penalty is 0.68/0.82, or 0.83. This implies the average distance traveled is 20% greater than the numerator ($\|\vec{\alpha}_{\omega \to CE}\| - \|\vec{\alpha}_{T \to CE}\|$) which is equivalent to the base measure estimate of 0.82. Therefore, the average distance traveled essentially matches that of the equilibrium path, matching the idea that the majority of the markets are finishing near a point on the contract curve.

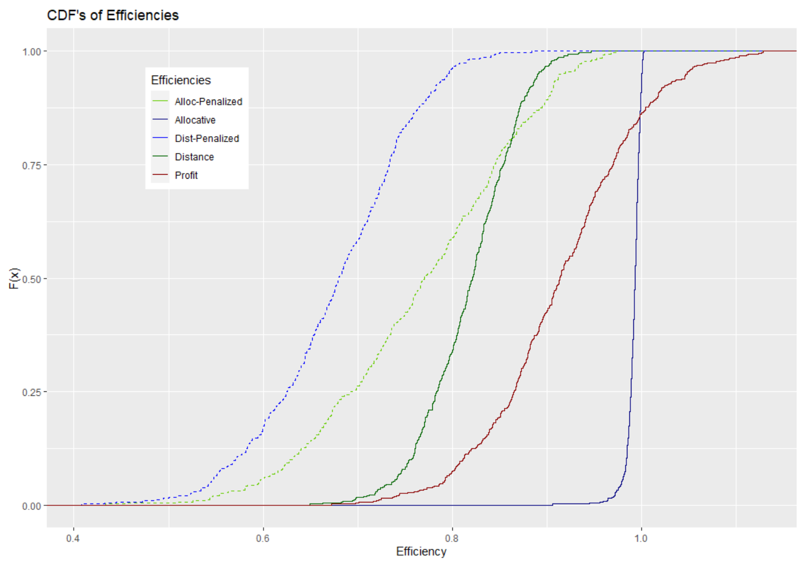

Figure 5: Cumulative Density Functions for Efficiency.

The final efficiency measure, profit efficiency, examines the market's ability to maximize gains from trade via price selection. As over-trading is possible in this model, the value can exceed 1. The average profit efficiency for the simulations is 0.91. Despite the ability to intentionally place utility-losing offers, the markets capture nearly all of the gains in the numeraire compared to equilibrium predictions. Sellers appear to be the group of traders comprimising on price more as they are capturing only 60% of their expected profit in Y, while buyers are overperforming with 124%. While pronounced, this difference is not particularly surprising as sellers' η is much larger throughout the period, leading them to post much more aggressive orders regardless of the side they enter. Additionally, the lower η of the buyers leads to a higher likelihood of auto-accepting those aggressive orders.

6 Concluding Remarks

This paper models market dynamics in an Edgeworth box where traders have 'imperfect' choice procedures when placing orders in a continuous double auction. Traders have the capacity to remember a portion of the history of the market, developing beliefs over the acceptability of order prices. Beliefs account for the relative success of each past price based on order size and fill. Agents recognize that they may participate on both sides of the market, and develop reservations depending on which side they enter. As traders maintain some utility preferences over their holdings, these reservations are held in terms of utility (as opposed to reservations on price as in Friedman (1991)). A curvature parameter η (which is a function of the time remaining in the market) determines what orders are immediately acceptable on the entered side and what orders satisfy the trader's reservation were they to enter on the contra-side in their next entry. Such a process allows traders to make order selections that appear to be utility-reducing relative to their true preferences, though allow the trader to position themselves as to better perform as a two-way trader.

A set of simulations test the performance of markets with computerized traders imbued with the behavior described in the model. Prices near the equilibrium prediction consistently. Round-averages remain slightly below equilibrium, creating tight bounds but not quite converging. Allocations, both in 16-space and 2-space, regularly lie in impressively close to Pareto optimal allocations along the contract curve at round's end, often very close to the equilibrium allocation bundle. Given rounds are not run intentionally until allocations are Pareto optimal, achieving nearly this so consistently is promising. Seller and buyer marginal rates of substitution provide supporting evidence for convergence in allocations as well. Efficiencies, both allocative and distance, are repeatedly high, suggesting gains from trade are often equitably spread and mostly drawn from the market.

The major implication of the findings of this project is the feasibility of boundedly rational order placement decisions in markets that show convergent tendencies. Specifically, strategic repositioning in the orderbook, and in anticipated holdings, is a legitimate consideration traders may be making in double auctions. This paper confirms such a consideration is not as harmful as some preferring perfectly rationality may suspect; in fact, estimates here perform near or level with some more complex models. Furthermore, the model provides a mapping from the zero intelligence gate (beginning with ZI) through the wilderness to a model fit much closer to the rational gate (this being Gjerstad and Dickhaut's belief-driven model).

A few natural adjustments to this model exist. First, individualized η functions, dependent on arguments such as current holdings, within-round and market-life earnings, and overall time in the market (aggregated across periods), is an interesting adaptation. Estimation of functional form for variations on η via laboratory experimentation could be illuminating for the external validity of this model's mechanism. Given the results of Williams (2021a), an inclusion of prices in the orderbook in the belief updating process would likely improve fit.

Declaration of Interest: none.

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Appendix A Robustness Checks

A.1 History Reset

	History Reset		
	Mean	St. Dev.	Range
I. Prices			
Price	2.48	0.16	(2.05, 3.04)
Per-Unit Avg.	2.37	0.16	(1.97, 2.78)
$\lvert Price - CE \rvert$	0.50	0.15	(0.23, 1.12)
RMSE	0.76	0.32	(0.31, 1.93)
Final 5 Prices	2.46	0.16	(2.07, 3.64)
II. Allocations			
Final Distance	0.63	0.33	(0.07, 1.80)
Seller MRS	2.44	0.13	(1.98, 3.00)
BuyerMRS	2.47	0.13	(2.13, 3.02)
III. Efficiencies			
Allocative	0.98	0.03	(0.74, 1.00)
- Penalized	0.82	0.09	(0.37, 0.97)
Distance	0.81	0.04	(0.61, 0.93)
- Penalized	0.67	0.08	(0.27, 0.84)
Profit	0.90	0.07	(0.54, 1.09)
Observations	240	240	240

Table A.1: Simulation outcomes for markets with the trader history reset at the beginning of every period.

Table A.1 shows simulation results for markets with trader's history entirely reset at the start of each period. As seen in the left panel, allowing memories to straddle periods is not driving the impressive results in the paper. In fact, resetting the history (and thus memories) each period yields slight improvements in most outcomes relative to the markets examined in the main text. Means for round-average prices and final prices fall just a few tenths above of the main simulations, though with tighter ranges. Measures of final distance and penalized allocative efficiency are just slightly improved in the markets with resetting memories; buyer and seller MRS actually shows a much tighter spread. Distance and profit efficiencies are essentially unchanged.

A.2 Other Memory Lengths

	L=0			L=10		
	Mean	St. Dev.	Range	Mean	St. Dev.	Range
I. Prices						
Prices	2.39	0.15	(2.01, 2.86)	2.57	0.17	(2.16, 3.12)
Per-Unit Avg.	2.31	0.16	(1.89, 2.76)	2.40	0.26	(1.76, 3.18)
$\|Price - CE\|$	0.45	0.12	(0.22, 0.96)	0.53	0.14	(0.24, 1.14)
RMSE	0.64	0.24	(0.27, 1.73)	0.93	0.30	(0.35, 1.94)
Final 5 Prices	2.45	0.15	(2.11, 3.11)	2.45	0.17	(1.93, 3.68)
II. Allocations						
Final Distance	0.66	0.35	(0.03, 2.10)	0.80	0.51	(0.00, 2.39)
Seller MRS	2.44	0.14	(2.04, 3.12)	2.43	0.14	(1.83, 2.72)
Buyer MRS	2.48	0.13	(2.09, 2.93)	2.50	0.16	(2.19, 3.19)
III. Efficiencies						
Allocative	0.97	0.03	(0.78, 1.00)	0.99	0.01	(0.96, 1.00)
-Penalized	0.82	0.09	(0.43, 0.97)	0.77	0.10	(0.51, 0.97)
Distance	0.81	0.05	(0.64, 0.92)	0.82	0.05	(0.69, 0.93)
-Penalized	0.67	0.09	(0.36, 0.85)	0.68	0.08	(0.43, 0.87)
Profit	0.90	0.07	(0.57, 1.06)	0.92	0.08	(0.63, 1.14)
Observations	240	240	240	240	240	240

Table A.2: Simulation outcomes for markets with memory lengths of 0 and 10 [20 runs each].

Table A.2 reports simulation results for markets with traders holding memories of L=0 and L=10. Given the tightness of the reservation utilities driven by the η in the main paper, results appear relatively robust for lower values of L (0, 5 or 10 here). L=10 reports mildly better estimates in prices (aside from RMSE), while L=0 narrowly nudges ahead in allocation outcomes. Efficiencies display a mild separation in average estimates in favor of L=10 markets and supports are tighter on nearly all measures. Relative to the L=5 estimates reported in Section 5.3, L=10 performs nearly identically, though with a negligible lead in

MRS estimates and slightly tighter supports across most estimates. L=0 seems to mildly outperform the L=5 allocation estimates, while systematically lower in prices.

A.3 Other λ levels

	$\lambda = 0$			$\lambda = 20$		
	Mean	St. Dev.	Range	Mean	St. Dev.	Range
I. Prices						
Price	2.78	0.42	(1.65, 4.65)	2.54	0.21	(0.96, 3.33)
Per-Unit Avg.	2.52	0.34	(1.55, 3.53)	2.46	0.24	(1.03, 3.18)
$\|Price - CE\|$	0.94	0.35	(0.34, 2.86)	0.37	0.17	(0.18, 1.48)
RMSE	1.40	0.60	(0.42, 3.67)	0.61	0.26	(0.25, 1.92)
Final 5 Prices	2.56	0.31	(1.87, 3.66)	2.44	0.17	(0.96, 3.07)
II. Allocations						
Final Distance	0.81	0.50	(0.01, 2.72)	0.80	0.79	(0.03, 6.63)
Seller MRS	2.36	0.19	(1.75, 3.02)	2.35	0.23	(0.90, 2.61)
Buyer MRS	2.55	0.21	(2.07, 3.34)	2.59	0.31	(2.29, 5.31)
III. Efficiencies						
Allocative	0.94	0.06	(0.44, 1.00)	0.99	0.07	(0.16, 1.00)
-Penalized	0.76	0.12	(0.15, 0.97)	0.87	0.11	(0.06, 0.98)
Distance	0.75	0.08	(0.39, 0.90)	0.86	0.08	(0.09, 0.95)
-Penalized	0.56	0.12	(0.11, 0.80)	0.76	0.10	(0.08, 0.91)
Profit	0.86	0.09	(0.15, 1.07)	0.92	0.08	(0.21, 1.08)
Observations	240	240	240	239	239	239

Table A.3: Simulation outcomes for markets with logit parameter, λ, values of 0 and 20. One round in the $\lambda = 20$ simulations had zero trades, hence the lower observation count.

The left panel of Table A.3 shows estimates for markets with a logit parameter of 0, meaning traders are placing uniform probability over their order choices. Unsurprisingly, nearly all estimates are worse than the $\lambda = 5$ markets reported in Section 5.3. What is surprising is how small some of the differences are. Buyer and Seller MRS still sit about 0.1 unit on either side of the CE price, though the supports suggest overtrading occurs more often when taking away the use for price beliefs. The deviations in efficiency estimates match are similar in magnitude to those of the L=0 markets in Appendix A.2; this suggests having no order history is as harmful to market performance as random selection. Price estimates show the largest deviation, with average price over 0.3 away from CE and some periods having an average nearly double the CE price.

The right panel shows estimates for markets with much more intelligent traders. With a logit parameter of $\lambda = 20$, small adjustments in expected utility across order options are more discernible to the trader, allowing for more precise order choice. While estimates in these markets do outperform the $\lambda = 0$ estimates, they do not outperform the main $\lambda = 5$ results. In some cases, estimates are marginally worse, in fact. Penalized distance efficiency is noticeably better (base Distance efficiency only marginally) in these more intelligent markets, while all Panel II estimates are quite close to the Table 1 estimates. Markets are thus arriving at final allocations a similar distance away from the CE bundle as the $\lambda = 5$ markets, however with these points lying closer to $\vec{v}_{\omega \to CE}$.

A.4 No Accept Rule

	$\lambda = 0$			$\lambda = 5$		
	Mean	St. Dev.	Range	Mean	St. Dev.	Range
I. Prices						
Price	2.42	0.43	(1.44, 3.62)	2.43	0.20	(1.83, 3.09)
Per-Unit Avg.	2.32	0.42	(1.39, 3.47)	2.35	0.27	(1.65, 3.07)
$\lvert Price - CE \rvert$	0.70	0.22	(0.22, 1.37)	0.45	0.17	(0.10, 1.23)
RMSE	0.87	0.27	(0.26, 1.65)	0.67	0.23	(0.12, 1.60)
Final 5 Prices	2.26	0.36	(1.45, 3.70)	2.43	0.12	(2.06, 2.87)
II. Allocations						
Final Distance	1.22	0.73	(0.05, 3.71)	0.81	0.52	(0.05, 3.47)
Seller MRS	2.25	0.28	(1.56, 2.95)	2.39	0.17	(1.63, 2.70)
Buyer MRS	2.70	0.33	(1.98, 3.76)	2.54	0.19	(2.27, 3.66)
III. Efficiencies						
Allocative	0.91	0.07	(0.59, 1.00)	0.99	0.01	(0.96, 1.00)
-Penalized	0.74	0.12	(0.33, 0.96)	0.82	0.09	(0.51, 0.99)
Distance	0.71	0.09	(0.36, 0.88)	0.85	0.05	(0.68, 0.94)
-Penalized	0.51	0.13	(0.14, 0.78)	0.72	0.08	(0.38, 0.88)
Profit	0.85	0.09	(0.51, 1.07)	0.95	0.08	(0.76, 1.15)
Observations	240	240	240	240	240	240

Table A.4: Simulation outcomes for markets with the accept rule removed from trader's decision process. Estimates are shown for markets with logit parameter values of 0 and 5.

Table A.4 reports simulation results for markets with traders who strictly place limit orders. This is more in-line with the majority of the trader behavior and market theoretical literature as limit orders are generally the only means of market participation in these simpler models. I test this simplification of the traders' order placement process in markets with logit parameters of 0 and 5.

First, in both the right and left panel, price-related estimates are systematically lower than the markets with the an accept rule. This is supported by (1) the lower expected CE price of 2.24 if traders only trade on their natural side while being guided by their $u_{i,-\Delta}$, and (2) the lack of an accept rule means more aggressive prices (those posted closer to the lower reservation IC) are not immediately taken. A similar explanation reconciles the improvement in profit efficiency in the $\lambda = 5$ panel.

The $\lambda = 0$ results are considerably lower than those of the $\lambda = 5$ markets. The largest difference appears in the final cluster of prices, with $\lambda = 0$ markets clearly tapering towards the $u_{i,-\Delta}$ CE price prediction, while $\lambda = 5$ markets make an effort to stay around the true CE prediction. Efficiency estimates are considerably lower regardless of measure for the $\lambda = 0$ sessions compared to both the right panel and the main results.

A.5 No Internal Spread Reduction Rule

	No Internal SR		
	Mean	St. Dev.	Range
I. Prices			
Price	2.56	0.28	(2.01, 4.07)
Per-Unit Avg.	2.35	0.29	(1.70, 3.67)
$\lvert Price - CE \rvert$	0.69	0.25	(0.28, 1.94)
RMSE	1.13	0.41	(0.43, 2.36)
Final 5 Prices	2.50	0.25	(2.05, 4.37)
II. Allocations			
Final Distance	0.87	0.49	(0.03, 2.41)
Seller MRS	2.48	0.13	(1.89, 2.78)
BuyerMRS	2.47	0.13	(2.14, 3.20)
III. Efficiencies			
Allocative	0.99	0.02	(0.68, 1.00)
- Penalized	0.76	0.11	(0.47, 0.96)
Distance	0.81	0.06	(0.49, 0.94)
- Penalized	0.67	0.09	(0.23, 0.85)
Profit	0.94	0.07	(0.61, 1.11)
Observations	240	240	240

Table A.5: Simulation outcomes for markets with no internal spread reduction rule imposed.

Table A.5 tests whether imposing an internal spread reduction rule impacts the performance of the model. An internal spread reduction rule restricts the trader from posting an order with a worse price than one she has already posted. This is different from the standard spread reduction rule, in which traders can only post orders that tighten the best bid-ask spread.

While price estimates appear to be negligibly worse than the main results, efficiency estimates are essentially equivalent, and MRS estimates converge (though slightly above CE). As the MRS estimates are measured for the representative agents, the improvement in convergence while maintaining the same efficiency levels points to slightly more trading across trader type (e.g., a trade between two natural sellers leads to no change in the representative sellers allocation).

Appendix B Only $u_{i,\Delta}$ Reservations

An interesting variation of the model presented in Sections 3 and 4 is one in which only the stricter reservation IC's are considered. This version of the model closely resembles what could be a GE-analog of the reservation price model proposed by Friedman (1991), though with the added intelligence of belief-formation on price acceptability and market orders. I'll refer to this variation as GE-F. Tables B.1 and B.2 show results for the GE-F simulated markets, with the latter excluding the acceptance rule step of order placement.

GE-F simulated markets with the accept rule report prices north of the true CE price of 2.44, and much closer to the price predicted for $u_{i,\Delta}$, 2.82. The $\lambda = 5$ batch is able to deviate away from this higher price to the true CE price thanks to the improved intelligence when making order choices. This is reflected as well in the tightening of the representative agent MRS estimates, though both batches are outperformed by the main simulations in Section 5.3. Average efficiency estimates in both distance measures and the penalized allocative measure for the $\lambda = 5$ batch outperform the main simulations.

Removing the accept rule displays an improvement in average prices, but a drastic decline in allocative performance. The distance and MRS estimates for the $\lambda = 0$ batch from Table B.2 suggest the traders' incapability to trade frequently once the only form of

	$\lambda = 0$			$\lambda = 5$		
	Mean	St. Dev.	Range	Mean	St. Dev.	Range
I. Prices						
Price	2.88	0.50	(1.34, 4.26)	2.70	0.38	(1.05, 3.71)
Per-Unit Avg.	2.70	0.35	(1.33, 3.78)	2.68	0.38	(1.06, 3.72)
$\lvert Price - CE \rvert$	0.89	0.40	(0.28, 2.41)	0.73	0.26	(0.26, 1.62)
RMSE	1.31	0.74	(0.36, 3.85)	1.18	0.51	(0.38, 2.82)
Final 5 Prices	2.72	0.43	(1.34, 4.84)	2.47	0.25	(1.05, 3.37)
II. Allocations						
Final Distance	1.25	0.86	(0.08, 6.43)	0.74	0.80	(0.03, 7.30)
Seller MRS	2.16	0.23	(0.95, 2.64)	2.29	0.20	(0.78, 2.53)
Buyer MRS	2.78	0.36	(2.18, 5.17)	2.61	0.36	(2.32, 5.95)
III. Efficiencies						
Allocative	0.91	0.10	(0.22, 0.99)	0.97	0.10	(0.02, 1.00)
-Penalized	0.80	0.12	(0.12, 0.97)	0.88	0.11	(0.01, 0.97)
Distance	0.73	0.11	(0.12, 0.87)	0.84	0.10	(0.01, 0.93)
-Penalized	0.57	0.14	(0.04, 0.81)	0.73	0.11	(0.00, 0.88)
Profit	0.85	0.10	(0.27, 1.00)	0.93	0.10	(0.02, 1.10)
Observations	240	240	240	238	238	238

Table B.1: Simulation outcomes for GE-Friedman markets with an accept rule.

trade is through posting crossing orders. This trend of minimal trading is made apparent by the lower end of the efficiency supports in all four batches of GE-F simulations. In fact, three of the four batches have at least one period with no trade whatsoever. The more restrictive reservations, without the more aggressive prices to grease the wheels of the market, have a difficult time both starting and continuing towards the equilibrium bundle. Increasing the logit parameter in the absence of an accept rule is able to make up for most of the gap in distance measures between $\lambda = 5$ GE-F markets with an accept rule and $\lambda = 0$ without one. Efficiency reacts similarly.

	$\lambda = 0$			$\lambda = 5$		
	Mean	St. Dev.	Range	Mean	St. Dev.	Range
I. Prices						
Price	2.46	0.37	(1.37, 3.45)	2.56	0.31	(1.13, 3.70)
Per-Unit Avg.	2.43	0.34	(1.43, 3.24)	2.55	0.32	(1.13, 3.53)
$\|Price - CE\|$	0.51	0.20	(0.15, 1.24)	0.53	0.21	(0.06, 1.65)
RMSE	0.62	0.23	(0.18, 1.45)	0.70	0.24	(0.08, 1.65)
Final 5 Prices	2.39	0.35	(1.37, 3.32)	2.43	0.24	(1.13, 3.38)
II. Allocations						
Final Distance	2.70	1.21	(0.34, 7.21)	1.15	1.02	(0.13, 7.28)
Seller MRS	1.79	0.28	(0.80, 2.61)	2.16	0.24	(0.79, 2.48)
Buyer MRS	3.37	0.52	(2.19, 5.87)	2.78	0.46	(2.38, 5.93)
III. Efficiencies						
Allocative	0.78	0.14	(0.04, 0.98)	0.95	0.13	(0.02, 1.00)
-Penalized	0.68	0.17	(0.02, 0.94)	0.90	0.14	(0.01, 0.99)
Distance	0.59	0.13	(0.02, 0.85)	0.82	0.13	(0.01, 0.94)
-Penalized	0.44	0.14	(0.00, 0.72)	0.75	0.14	(0.00, 0.91)
Profit	0.73	0.13	(0.05, 0.93)	0.90	0.12	(0.03, 1.09)
Observations	239	239	239	239	239	239

Table B.2: Simulation outcomes for GE-Friedman markets without the acceptance step of trader's order placement process.